On two-sided gamma-positivity for simple permutations

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joint work with
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The Eulerian numbers

- The Eulerian number $a_{n,k}$ counts the number of permutations in S_n, having $\mathrm{des}(\pi) = k$, where $\mathrm{des}(\pi)$ is the number of descents of a permutation $\pi \in S_n$:

Example

- $\pi_1 = 123, des(\pi_1) = 0$
- $\pi_2 = 132, des(\pi_2) = 1,$
- $\pi_3 = 213, des(\pi_3) = 1$
- $\pi_4 = 231, des(\pi_4) = 1$
- $\pi_5 = 312, des(\pi_5) = 1$
- $\pi_6 = 321, des(\pi_6) = 2$
- $a_{3,0} = 1, a_{3,1} = 4, a_{3,2} = 1$

The Eulerian polynomial

- The (one-sided) *Eulerian polynomial* is

$$A_n(q) = \sum_{\pi \in S_n} q^{\mathrm{des}(\pi)} = \sum_{k=0}^{n-1} a_{n,k} q^k$$

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Example

$A_2(q) = 1 + q$
$A_3(q) = 1 + 4q + q^2$
$A_4(q) = 1 + 11q + 11q^2 + q^3$

Palindromic polynomials

Definition

A polynomial $f(q) = a_r q^r + a_{r+1} q^{r+1} + \cdots + a_s q^s$ is **palindromic** if its coefficients are the same when read from left to right as from right to left.
equivalently, $f(q) = q^{r+s} f(1/q)$.
we define the **darga** of $f(q)$ as above to be $r + s$

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$A_2(q) = 1 + q$ is a palindromic of darga 1
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Theorem

The (one-sided) Eulerian polynomial

$$A_n(q) = \sum_{k=0}^{n-1} a_{n,k} q^k$$

is palindromic of darga $n-1$

The Gamma Basis

Theorem

The set of palindromic polynomials of darga $n-1$ is a vector space of dimension $\lfloor (n+1)/2 \rfloor$, with **gamma basis:**

$$\Gamma_{n-1} = \{q^j(1+q)^{n-1-2j} \mid 0 \leq j \leq \lfloor (n-1)/2 \rfloor\}.$$

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Thus there are real numbers $\gamma_{n,j}$ such that

$$A_n(q) = \sum_{0 \le j \le \lfloor (n-1)/2 \rfloor} \gamma_{n,j} q^j (1+q)^{n-1-2j}.$$

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- A combinatorial proof, based on an action called 'valley hopping' which has its roots in the work of Foata and Strehl from 1974, was re-discovered by Shapiro, Woan, and Getu in 1983, and was dusted off more recently by Branden in 2008.

Two-sided Eulerian Polynomial

Definition

Let $A_n(s,t)$ be the *two-sided Eulerian polynomial*

$$A_n(s,t) = \sum_{\pi \in S_n} s^{\mathrm{des}(\pi)} t^{\mathrm{ides}(\pi)}.$$

where $\mathrm{ides}(\pi) = \mathrm{des}(\pi^{-1})$

Two-sided Eulerian Polynomial

Example

The two-sided Eulerian polynomial for S_4 is:

$$A_4(s,t) = 1 + 10st + 10(st)^2 + (st)^3 + st^2 + s^2t.$$

Its matrix of coefficients is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 10 & 1 & 0 \\ 0 & 1 & 10 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

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A bivariate polynomial is **palindromic of darga** $n-1$ if its $n \times n$ matrix of coefficients is symmetric with respect to both diagonals.

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The set of bivariate palindromic polynomials of darga $n-1$ is a vector space with **bivariate gamma basis**

$$\Gamma_{n-1} = \{(st)^i (s+t)^j (1+st)^{n-1-j-2i} \mid i,j \geq 0,\ 2i+j \leq n-1\}.$$

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Example

$A_3(s,t) = (1+st)^2 + 2st$
$A_4(s,t) = (1+st)^3 + 7st(1+st) + st(s+t)$

Two sided gamma-positivity

Theorem (Gessel's conjecture, Lin's theorem)

For each $n \geq 1$ *there exist* **nonnegative integers** $\gamma_{n,i,j}$ $(i, j \geq 0,\ 2i + j \leq n - 1)$ *such that*

$$A_n(s,t) = \sum_{i,j} \gamma_{n,i,j} (st)^i (s+t)^j (1+st)^{n-1-j-2i}.$$

No combinatorial proof of Gessel's conjecture is known.

Simple permutations

Definition

Let $\pi = a_1 \dots a_n \in S_n$. A *block* (or *interval*) of π is a nonempty contiguous sequence of entries $a_i a_{i+1} \dots a_{i+k}$ whose values also form a contiguous sequence of integers.

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Example

If $\pi = 2647513$ then 6475 is a block but 64751 is not.

Simple permutations

Each permutation can be decomposed into singleton blocks, and also forms a single block by itself. These are the *trivial blocks* of the permutation. All other blocks are called *proper*.

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Example

- The simple permutation of order 1 is 1
- The simple permutations of order 2 are: 12 and 21
- There are no simple permutations of order 3.
- The simple permutations of order 4 are: 2413 3142

Example

The permutation 3517246 is simple.

Two-sided Eulerian Polynomial for simple permutations

Definition (A.B.E.R.S.)

For each positive integer n, define the **two-sided Eulerian Polynomial for simple permutations**

$$simp_n(s,t) = \sum_{\sigma \in Simp_n} s^{\mathrm{des}(\sigma)} t^{\mathrm{ides}(\sigma)}$$

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- $simp_n(s,t)$ is palindromic of darga $n-1$.
- Therefore it has a representation as a linear combination of the gamma basis, i.e., there exit real numbers $\gamma_{n,i,j}$ such that

$$simp_n(s,t) = \sum_{Simp_n} \gamma_{n,i,j}(st)^i (s+t)^j (1+st)^{n-1-j-2i}.$$

Conjecture Adin,Bagno,Eisenberg,R,S

Conjecture [A.B.E.R.S]: The coefficients of the polynomial are nonnegative integers:

For each $n \geq 1$ there exist **nonnegative integers** $\gamma_{n,i,j}$ $(i, j \geq 0,\ 2i + j \leq n - 1)$ such that

$$simp(s, t) = \sum_{Simp_n} \gamma_{n,i,j}(st)^i(s + t)^j(1 + st)^{n-1-j-2i}.$$

Two-sided Eulerian Polynomial for simple permutations

- $simp_1(s,t) = 1$.
- $simp_2(s,t) = 1 + st$.
- $simp_4(s,t) = s^2t + st^2 = st(s+t)$.
- $simp_5(s,t) = 6(st)^2$.
- $simp_6(s,t) = st(s+t)^2(1+st) + 5(st)^2(1+st) + 14(st)^2(s+t)$

In fact, our conjecture has been verified by computer for all $n \leq 12$.

Review

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Review

- Gessel's conjecture: The two sided Eulerian polynomials are gamma positive.
- It was proved by Lin, but there is no combinatorial proof.
- Definition (ABERS): Two sided Eulerian polynomials for simple permutations
- Conjecture (ABERS): The two sided Eulerian polynomials for simple permutations are gamma positive.
- Provided the settlement of our conjecture, we present a combinatorial proof of Lin's Theorem.

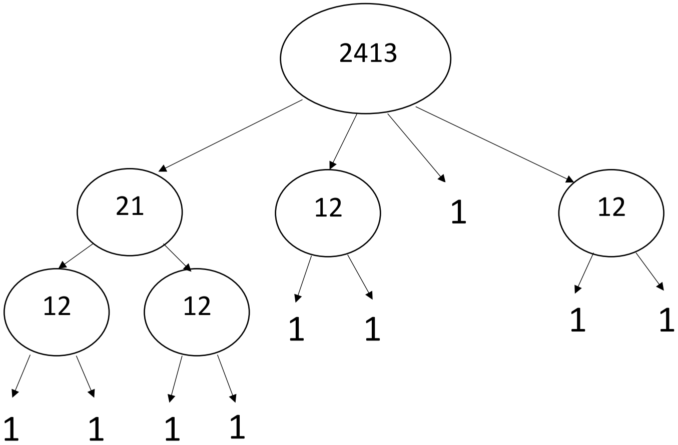
Decomposition of a permutation

Example

$$4523\ 98\ 1\ 67 = 2413[3412, 21, 1, 12].$$

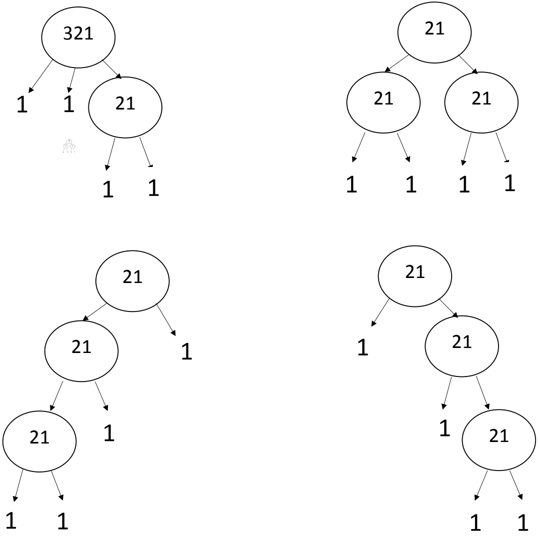
note that 3412 is not simple, and we can write $3412 = 21[12, 12]$

The tree of $\sigma = 452398167 = 2413[21[12, 12], 21, 1, 12]$



Uniqueness?

Trees for the permutation $\sigma = 4321$



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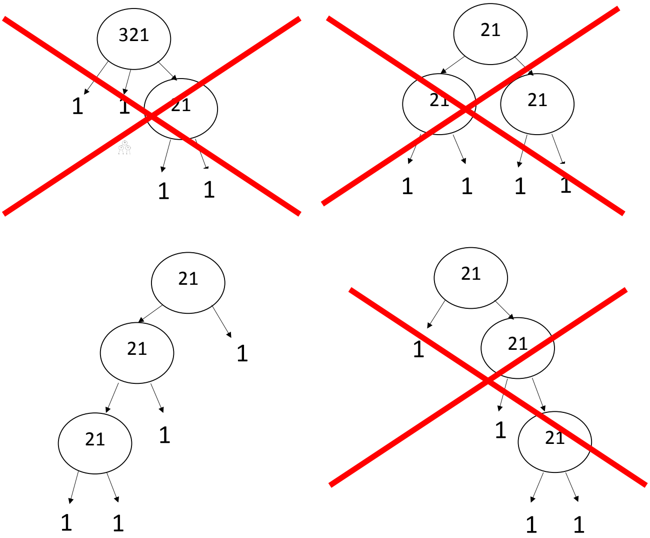


G-tree

Definition

A tree T is called a *G-tree* if it satisfies:

1. Each leaf is labeled by 1.
2. Each internal node is labeled by a simple permutation ($\neq 1$), and the number of its children is equal to the length of the permutation.
3. The labels in each binary right chain (BRC) alternate between 12 and 21.

Denote by $\mathcal{GT}_n$ the set of all G-trees with n leaves.

Every permutation has a unique G-tree.

The des and ides of inflation

Lemma [A.B.E.R.S.]:
Let $\sigma = \pi[\alpha_1, \ldots, \alpha_k]$. Then

$$\mathrm{des}(\sigma) = \mathrm{des}(\pi) + \sum_{i=1}^{n} \mathrm{des}(\alpha_i)$$

and

$$\mathrm{ides}(\sigma) = \mathrm{ides}(\pi) + \sum_{i=1}^{n} \mathrm{ides}(\alpha_i)$$

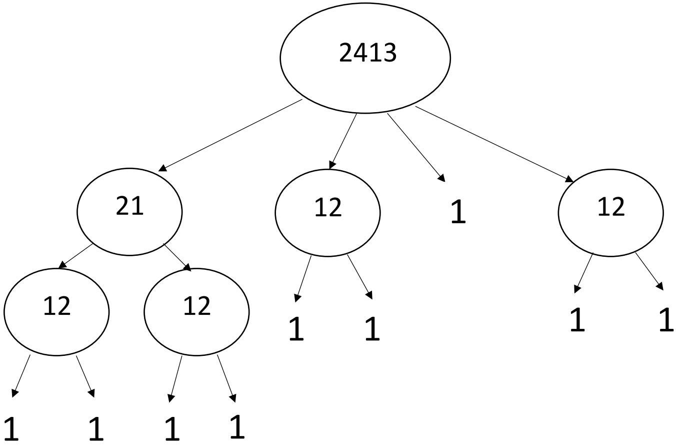
therefore

$$s^{\mathrm{des}(\sigma)} t^{\mathrm{ides}(\sigma)} = s^{\mathrm{des}(\pi)} t^{\mathrm{ides}(\pi)} \prod_{i=1}^{n} s^{\mathrm{des}(\alpha_i)} t^{\mathrm{ides}(\alpha_i)}$$

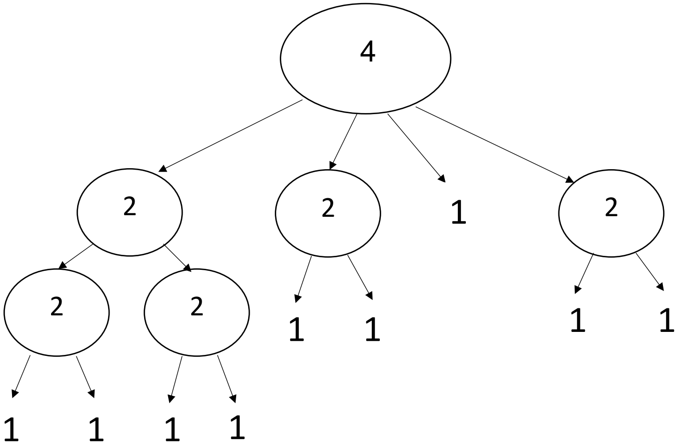
Two steps of the combinatorial proof for Lin's Theorem

- Divide the set of trees into equivalence classes.
- Prove that the Eulerian polynomial of each class is gamma positive (based on our conjecture regarding gamma positivity of simple permutations).

The tree for $\sigma = 452398167$



Simplified tree

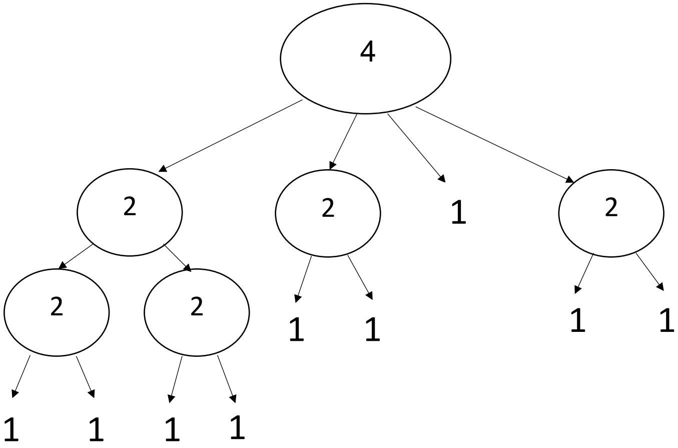


Equivalence classes of trees

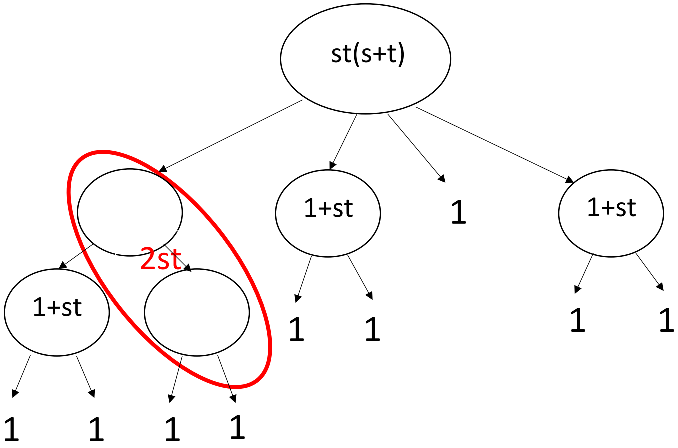
Definition

For permutations $\sigma_1, \sigma_2 \in S_n$ define $\sigma_1 \sim \sigma_2$ if $T'_{\sigma_1} = T'_{\sigma_2}$. Clearly $\sim$ is an equivalence relation on S_n, with each equivalence class corresponding to a unique simplified tree T'. Denote such a class by $A(T')$.
(With the single restriction that the labels in each BRC must alternate between 12 and 21, starting with either of them.)

Simplified tree



The polynomial of this simplified tree



The polynomial of a simplified tree

It thus follows that for each simplified tree T', the polynomial

$$\sum_{\sigma \in A(T')} s^{\mathrm{des}(\sigma)} t^{\mathrm{ides}(\sigma)}$$

is a product of factors, as follows:

- Each internal node with label $k \geq 4$ contributes a factor $simp_k(s, t)$.
- Each BRC of even length $2k$ contributes a factor $2(st)^k$.
- Each BRC of odd length $2k+1$ contributes a factor $(st)^k(1+st)$.

By our conjecture, all those factors are gamma-positive, and so is their product. Summing over all equivalence classes in S_n completes the combinatorial proof for Lin's Theorem.

Thank you!