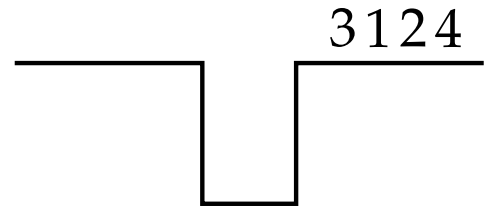
Enumerating permutations sortable by k passes through a pop-stack

Anders Claesson
Bjarki Ágúst Guðmundsson
University of Iceland

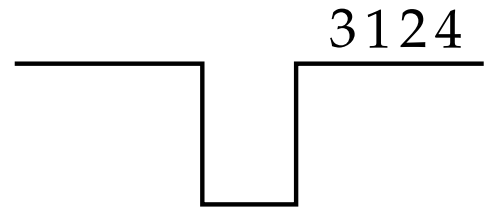
Theorem (Knuth, 1968)

A permutation π *is sortable by a stack if and only if* π *avoids* 231.



Theorem (Avis & Newborn, 1981)

A permutation π *is sortable by a pop-stack if and only if* π *avoids* 231 *and* 321.



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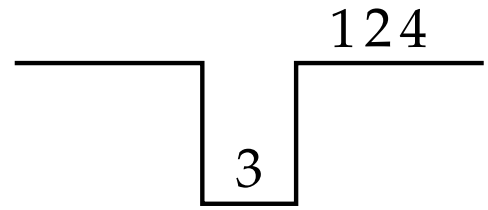

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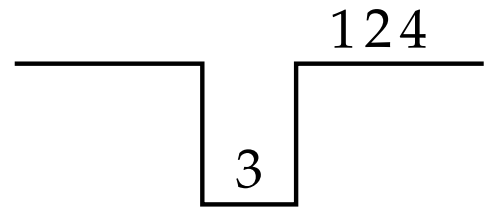

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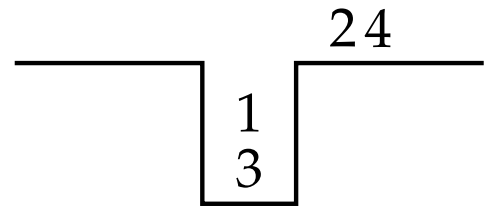

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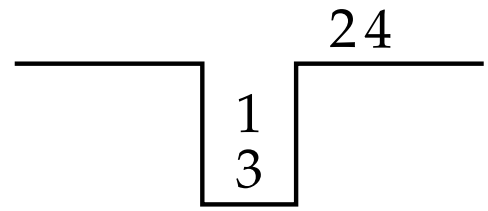

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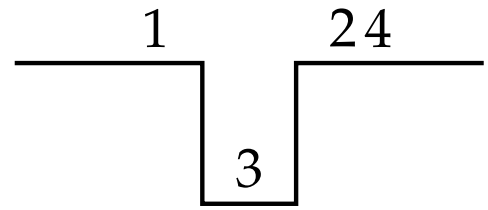

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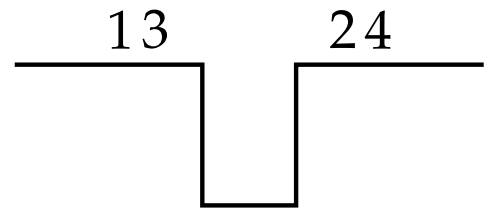

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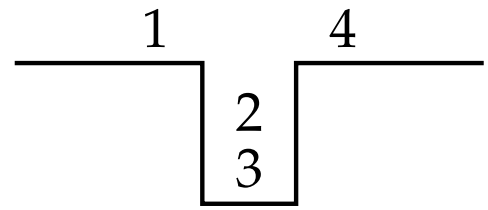

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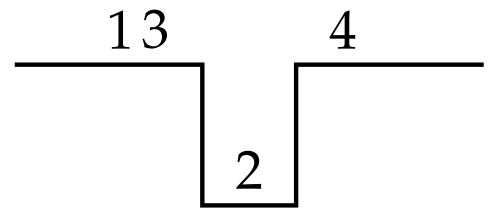

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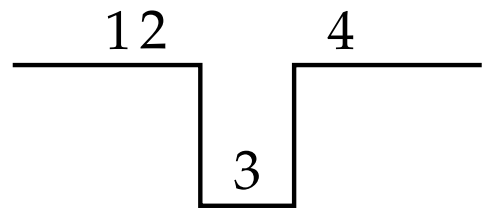

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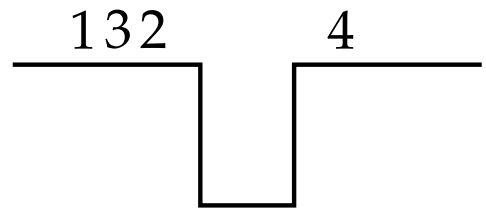

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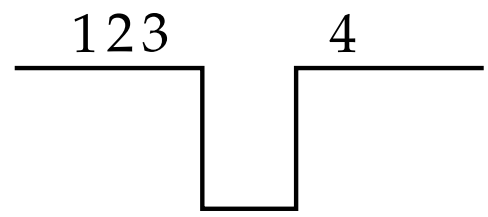

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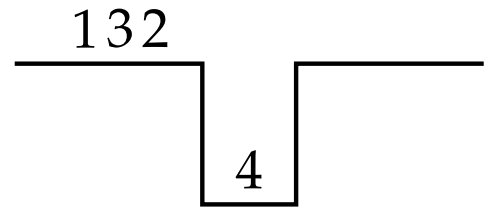

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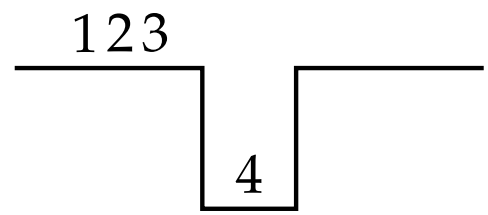

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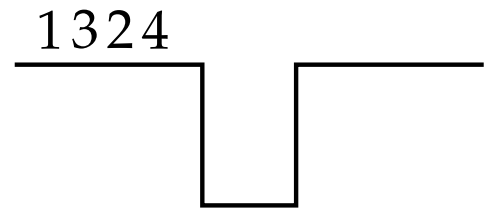

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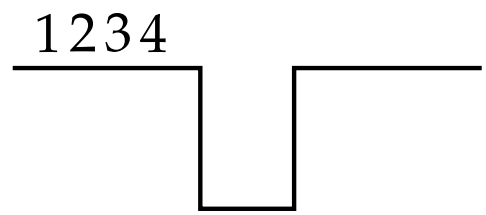

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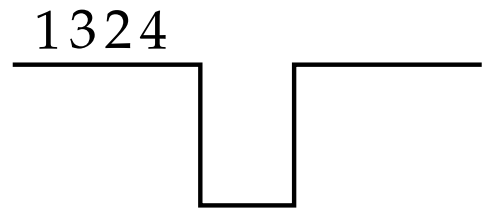

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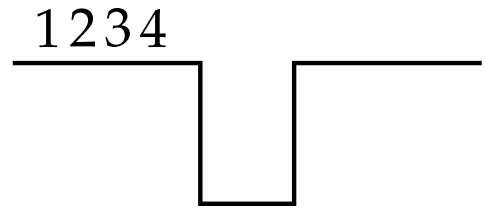

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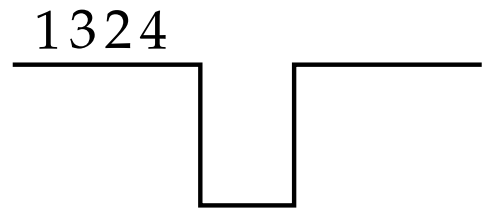

Theorem (Pudwell & Smith, 2017)

A permutation π *is sortable by two passes through a pop-stack if and only if* π *avoids* 2341, 3412, 3421, 4123, 4231, 4312, $4\bar{1}352$ *and* $413\bar{5}2$.

Theorem (Knuth, 1968)

A permutation π is sortable by a stack if and only if π avoids 231.

$$\frac{1-\sqrt{1-4x}}{2x}$$

Theorem (Avis & Newborn, 1981)

A permutation π is sortable by a pop-stack if and only if π avoids 231 *and* 321.

$$\frac{x-1}{2x-1}$$

Theorem (Pudwell & Smith, 2017)

A permutation π is sortable by two passes through a pop-stack if and only if π avoids 2341, 3412, 3421, 4123, 4231, 4312, $4\bar{1}352$ *and* $413\bar{5}2$.

$$\frac{x^3+x^2+x-1}{2x^3+x^2+2x-1}$$

Stacks vs pop-stacks

A stack is a LIFO data structure with two operations:

- Push: Add an element to the top of the stack.
- Pop: Remove the top-most element from the stack.

A pop-stack is a LIFO data structure with two operations:

- Push: Add an element to the top of the stack.
- Pop: Remove all elements from the stack.

We'll insist that elements on the stack are increasing when read from top to bottom and sort greedily w.r.t. the push operation.

How many permutations are sortable by k passes through a pop-stack?

$k = 1$	$\dfrac{x-1}{2x-1}$
$k = 2$	$\dfrac{x^3+x^2+x-1}{2x^3+x^2+2x-1}$
$k \geq 3$	Rational?

5 1 2 4 7 8 6 3 9


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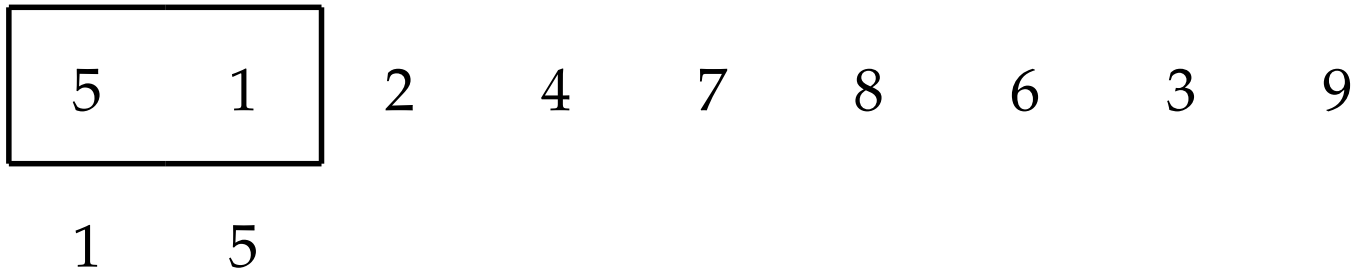
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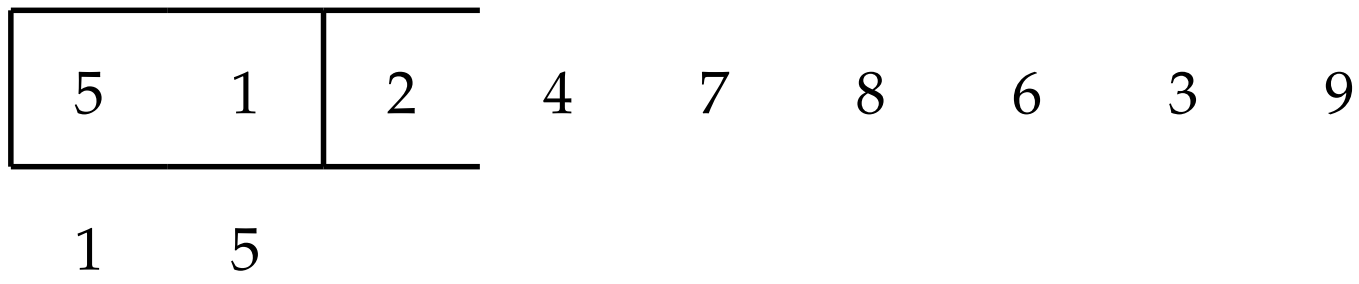
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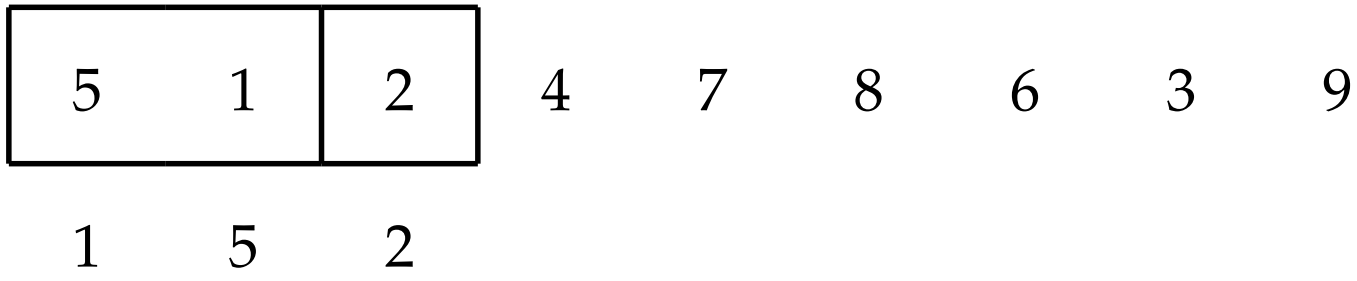
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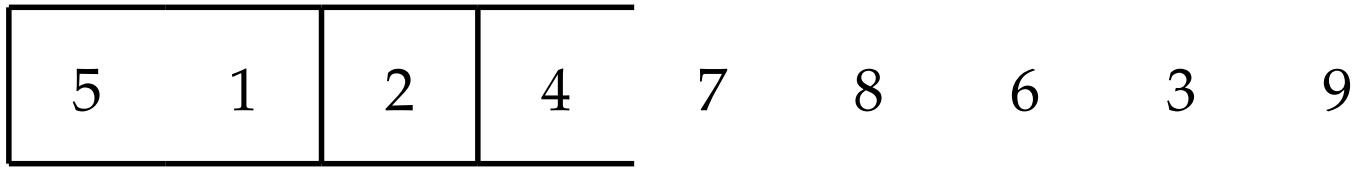
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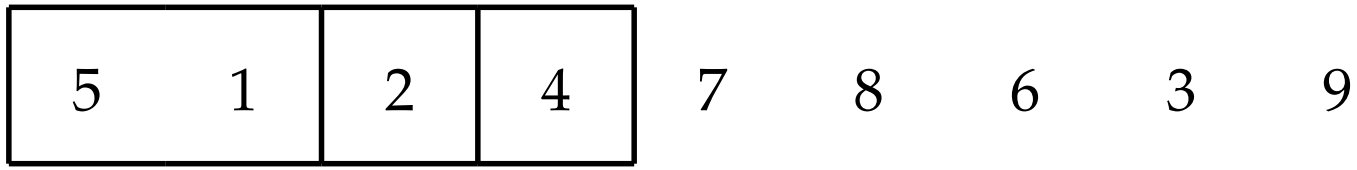
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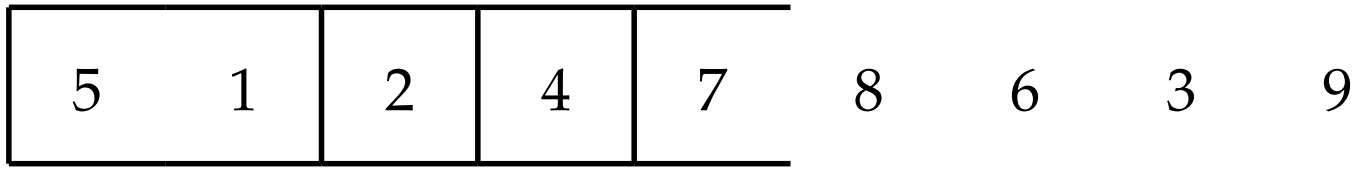

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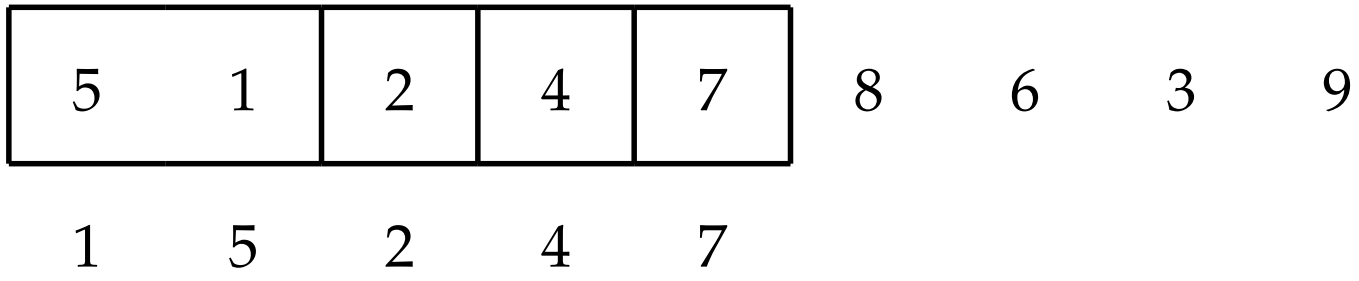
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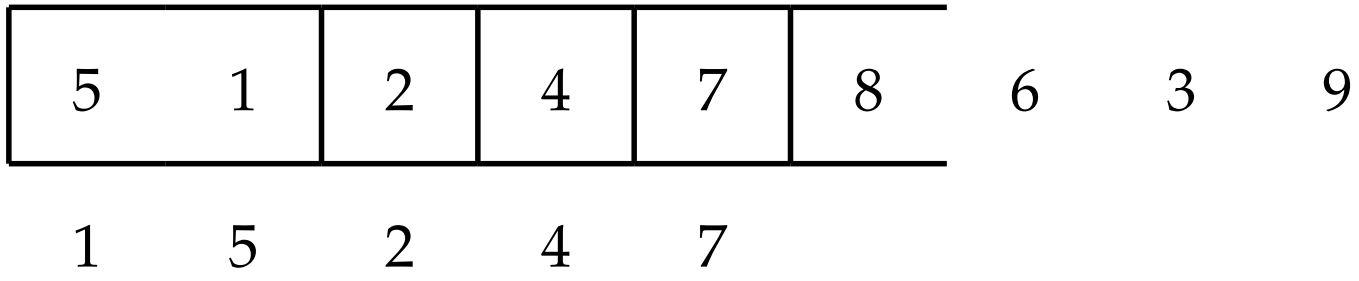
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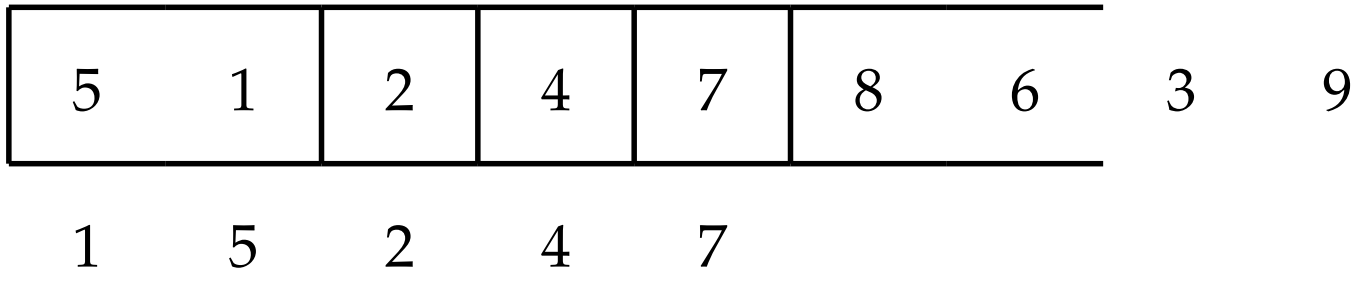
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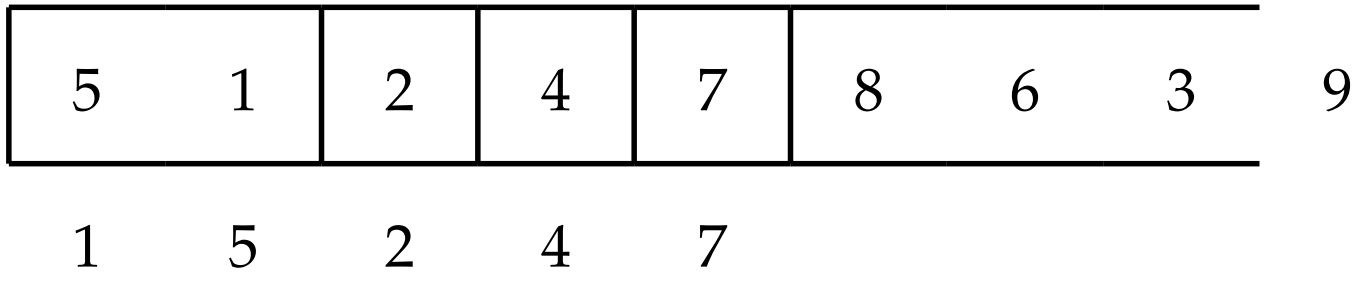
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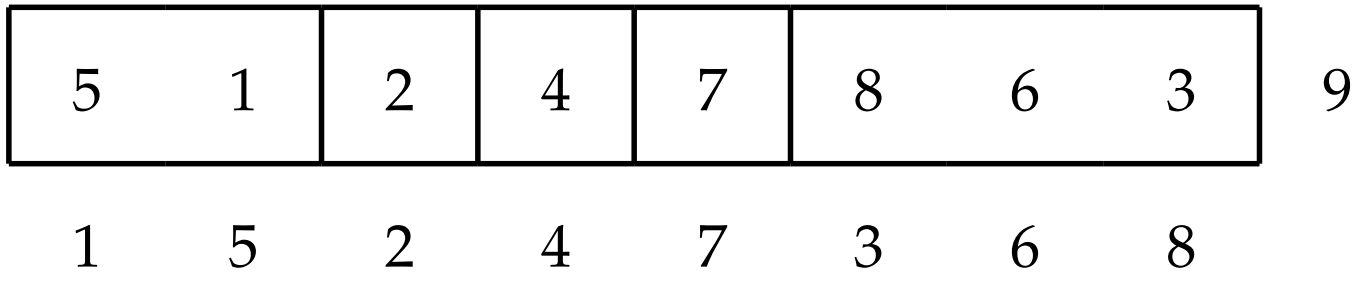
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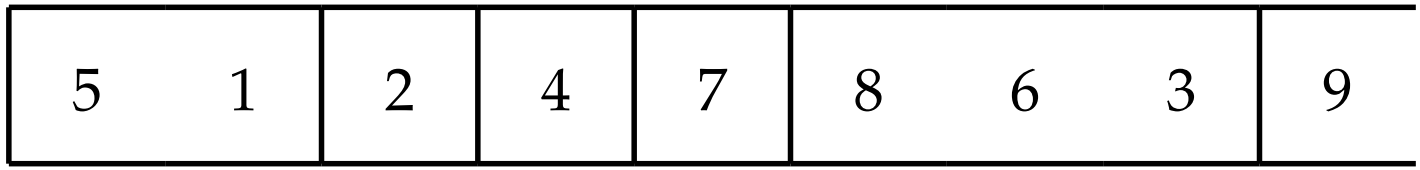

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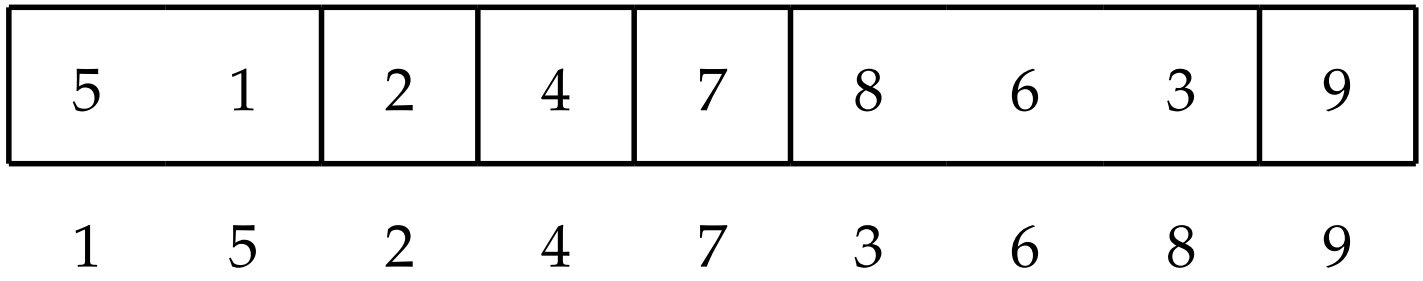
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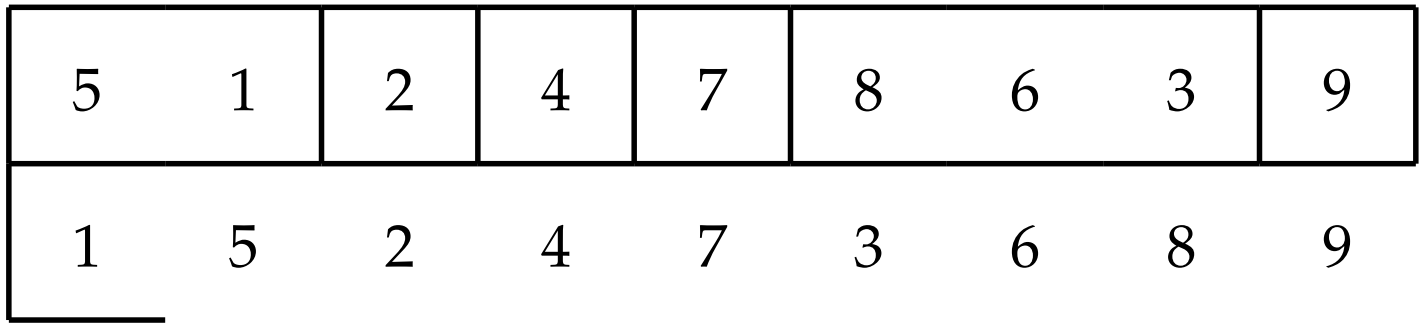
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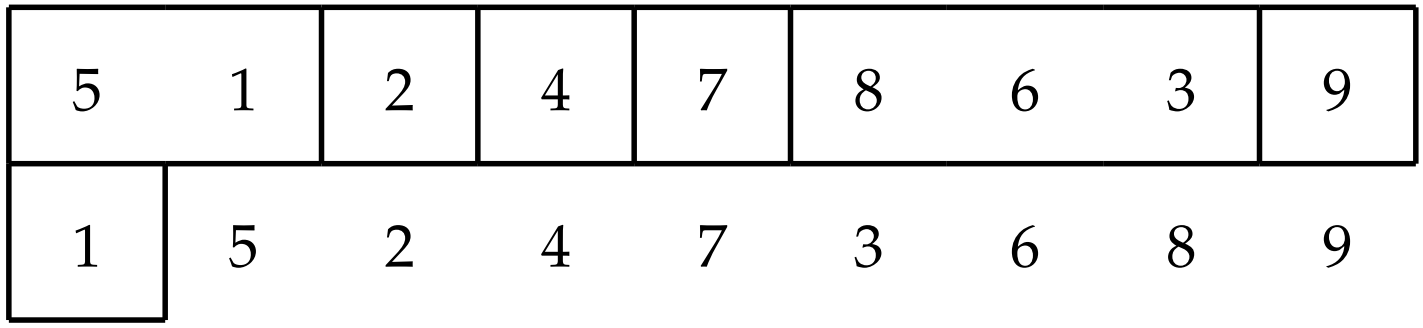

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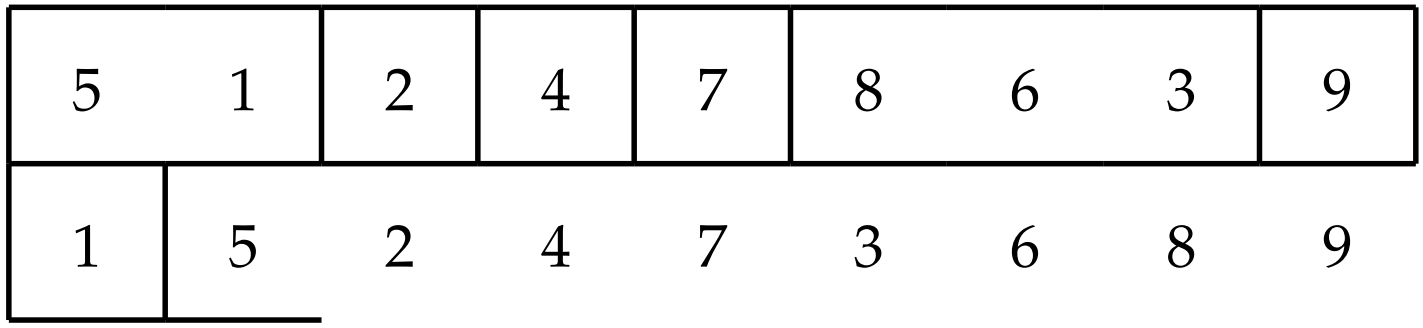

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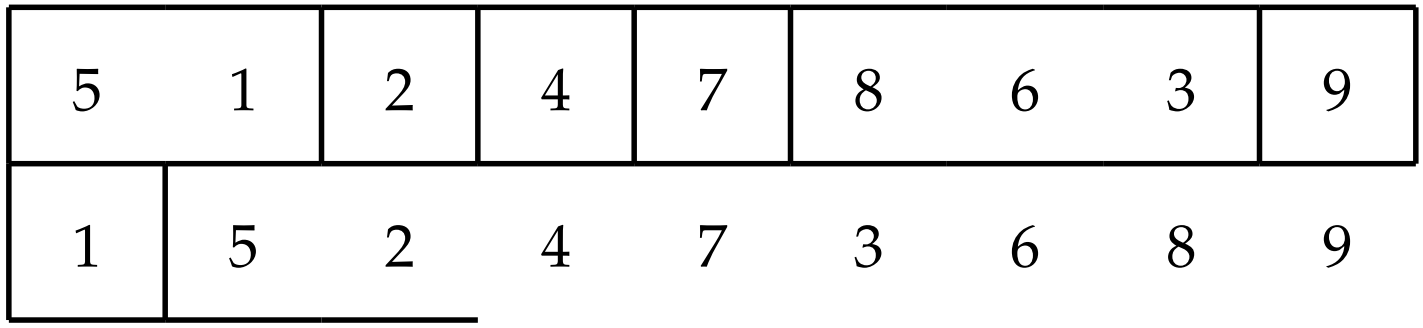

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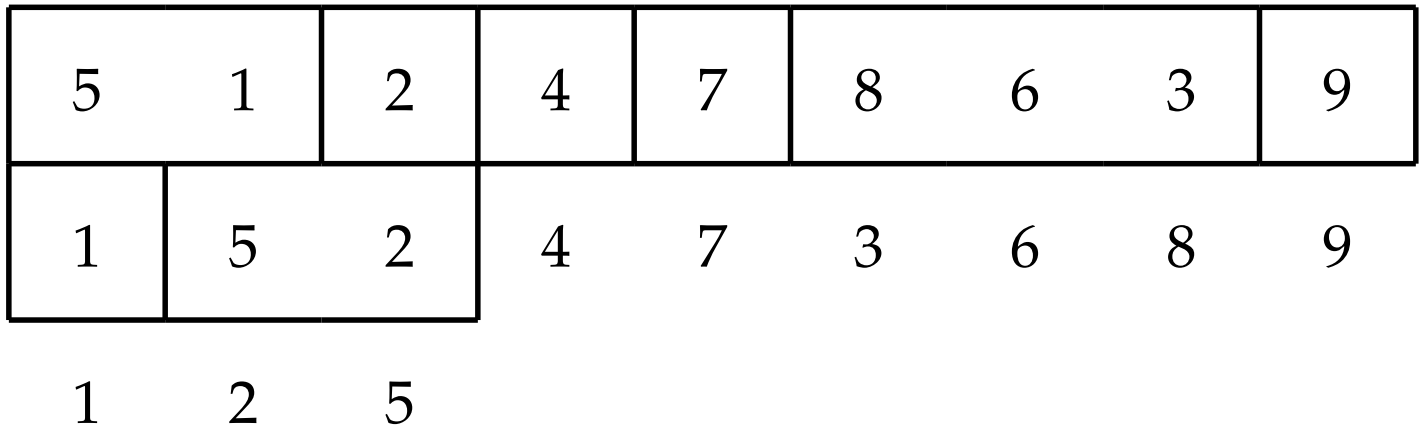
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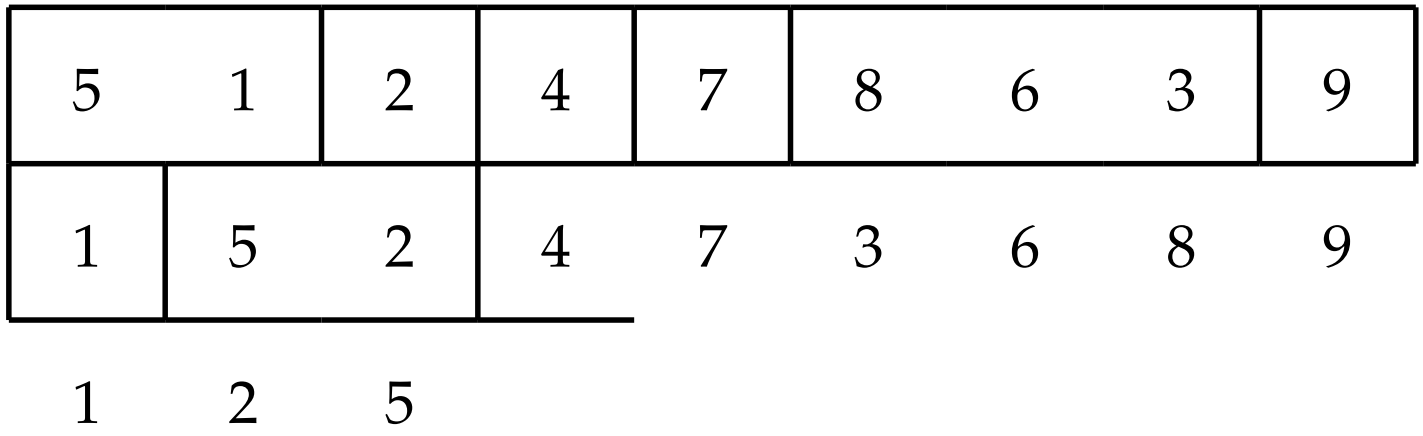
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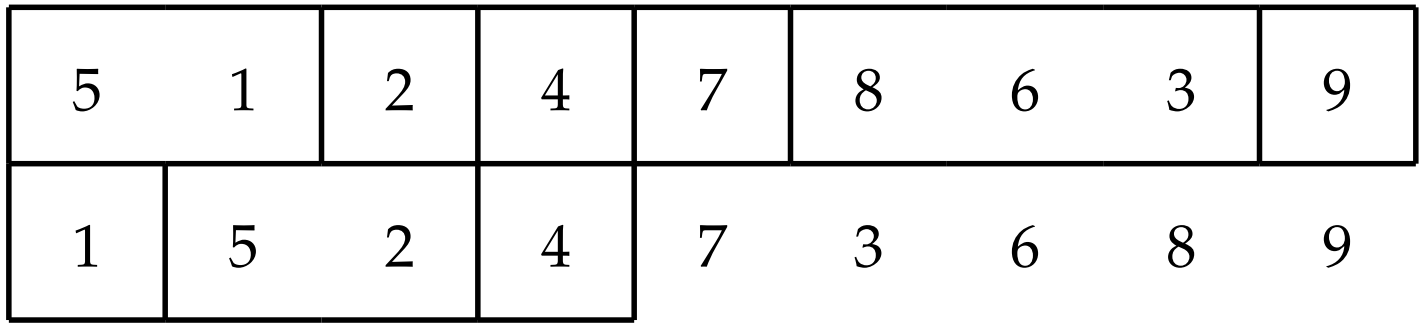
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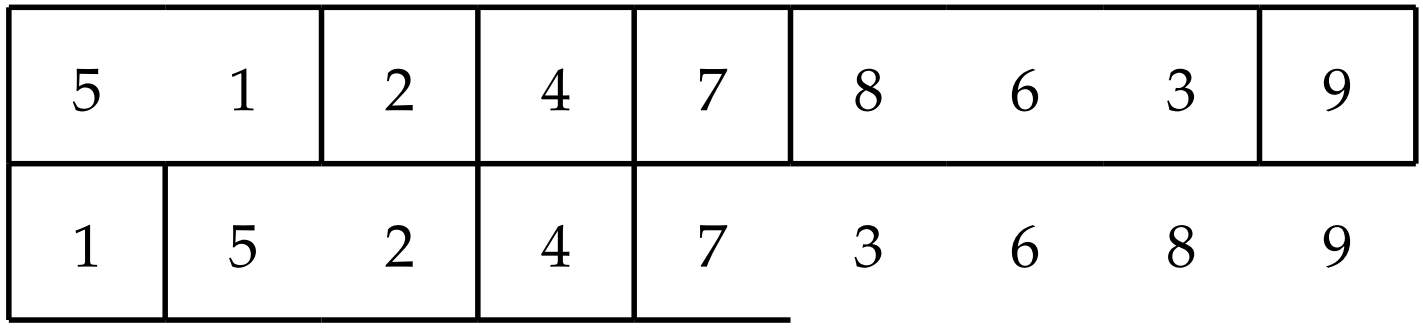

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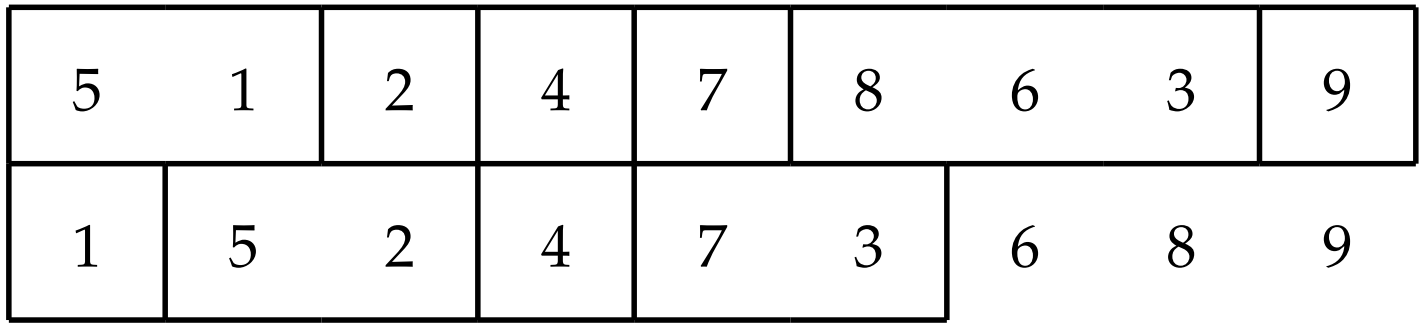
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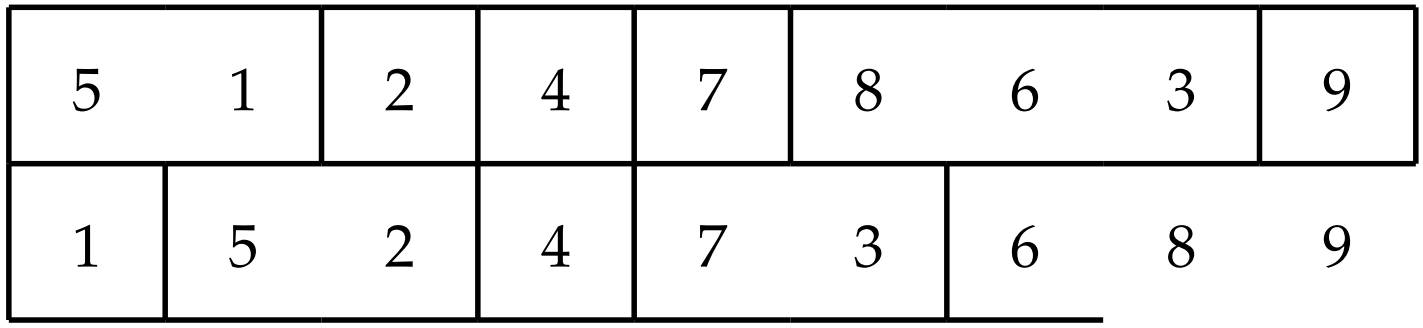

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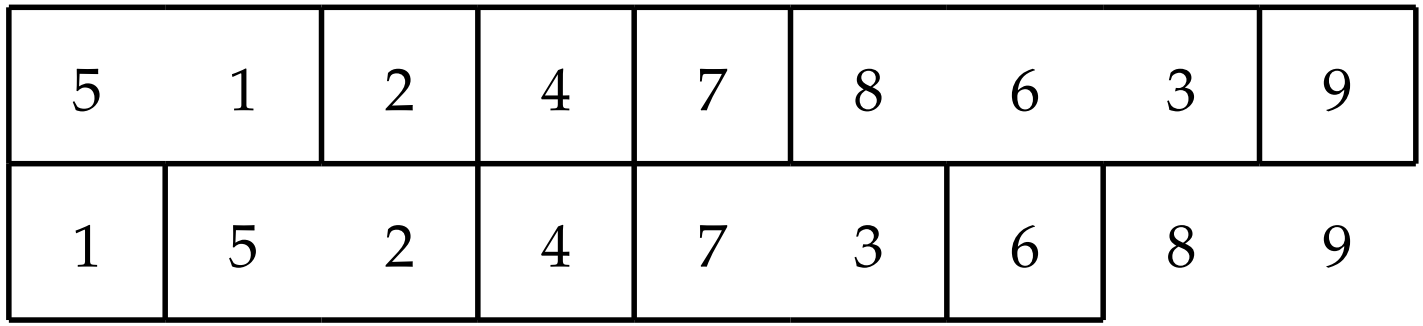
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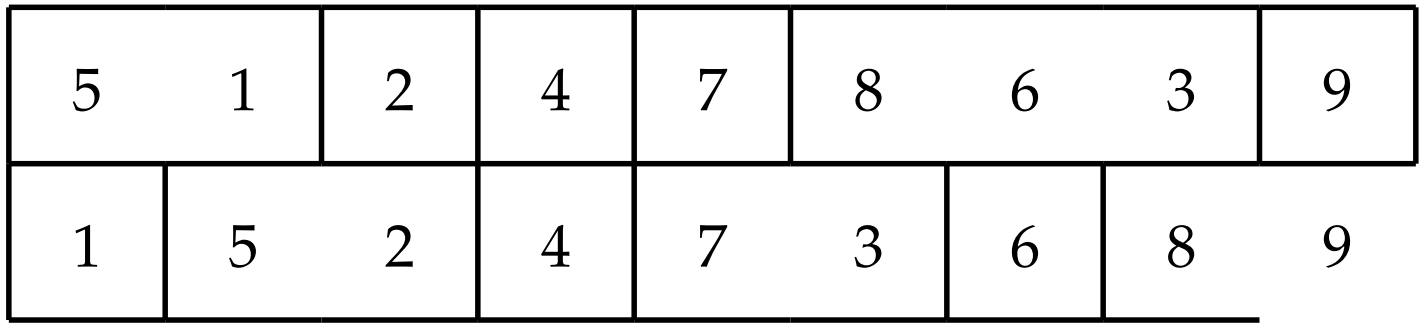

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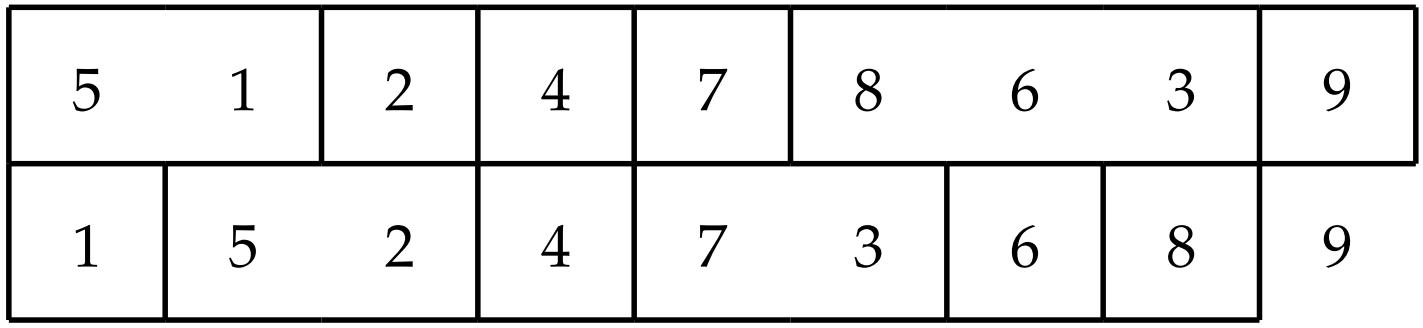
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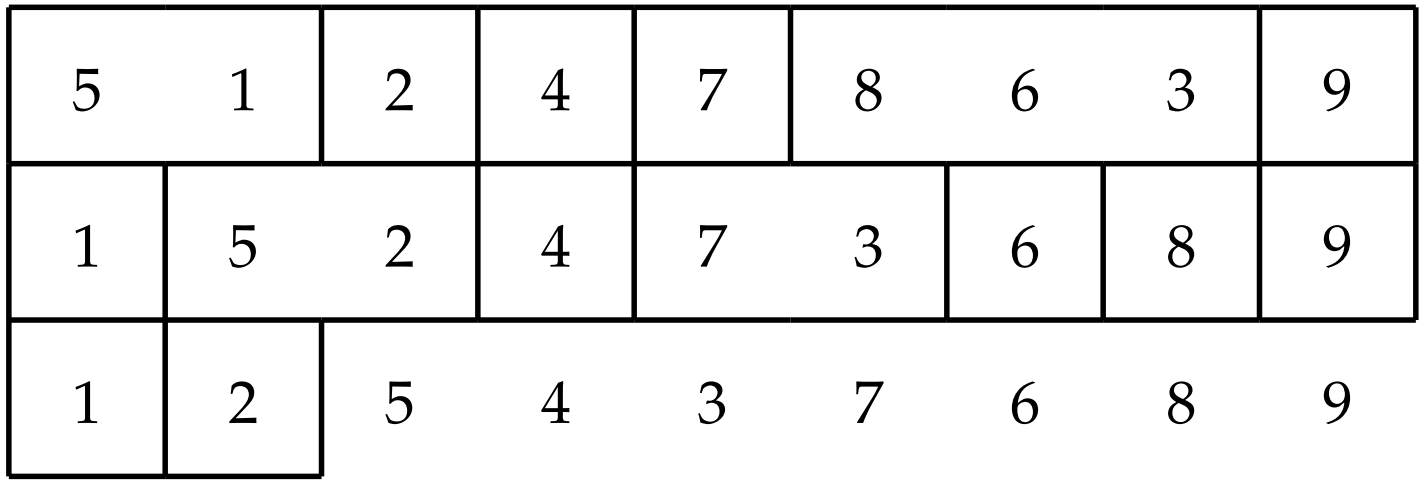
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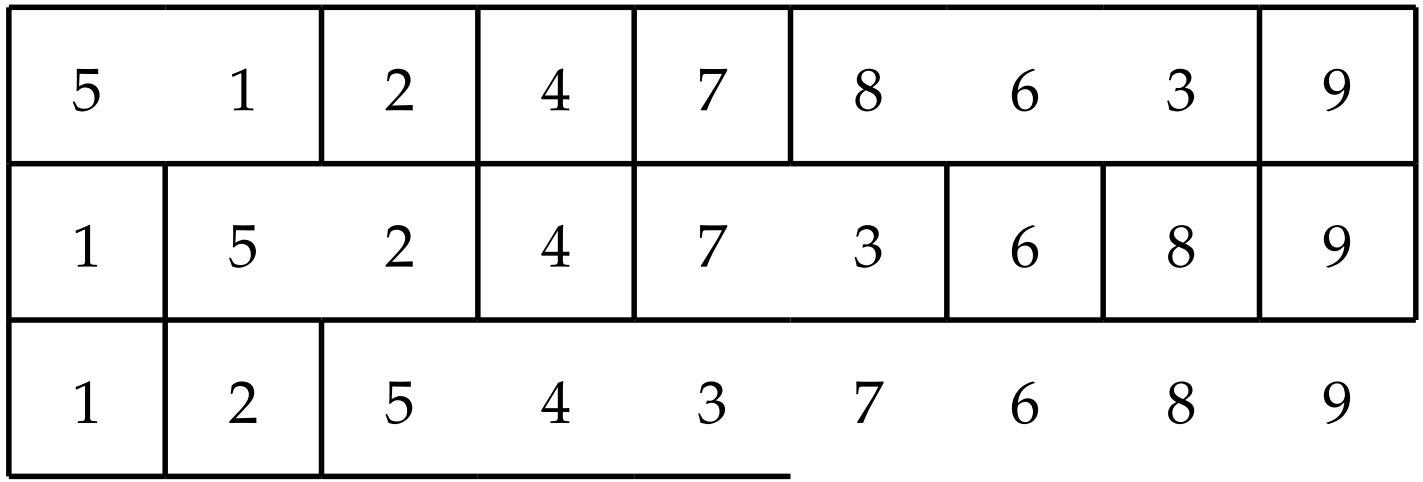
5	1	2	4	7	8	6	3	9
1	5	2	4	7	3	6	8	9
1	2	5	4	3	7	6	8	9

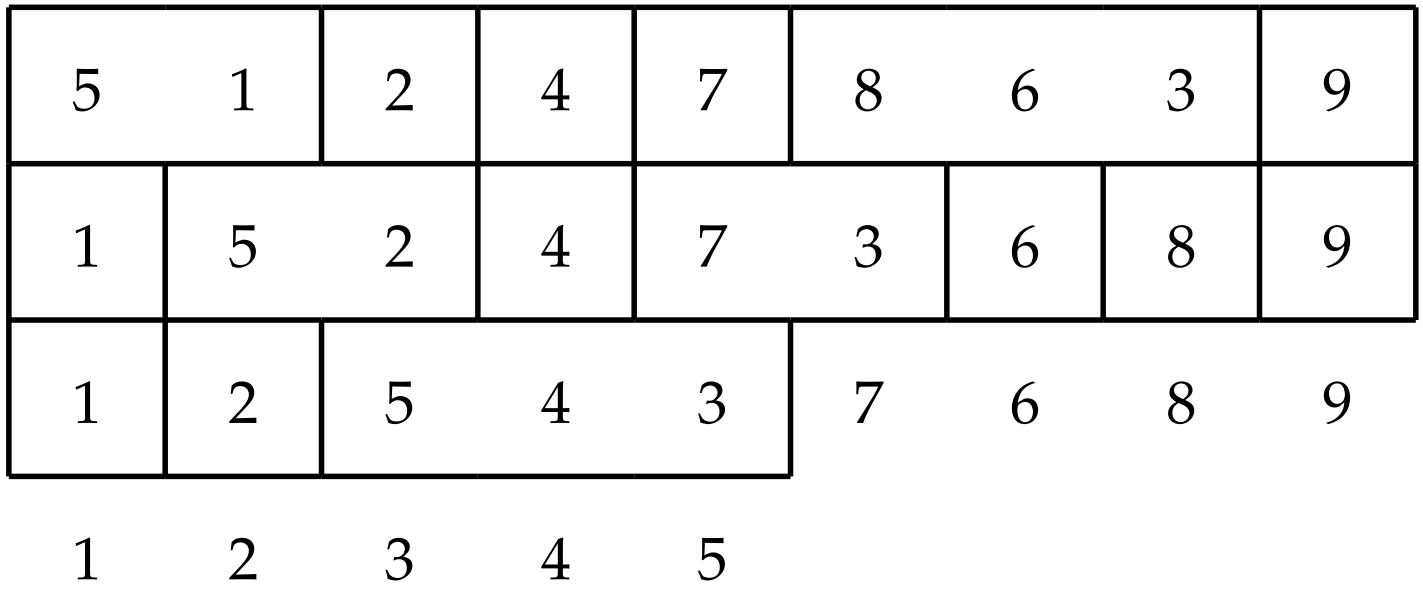
1 2

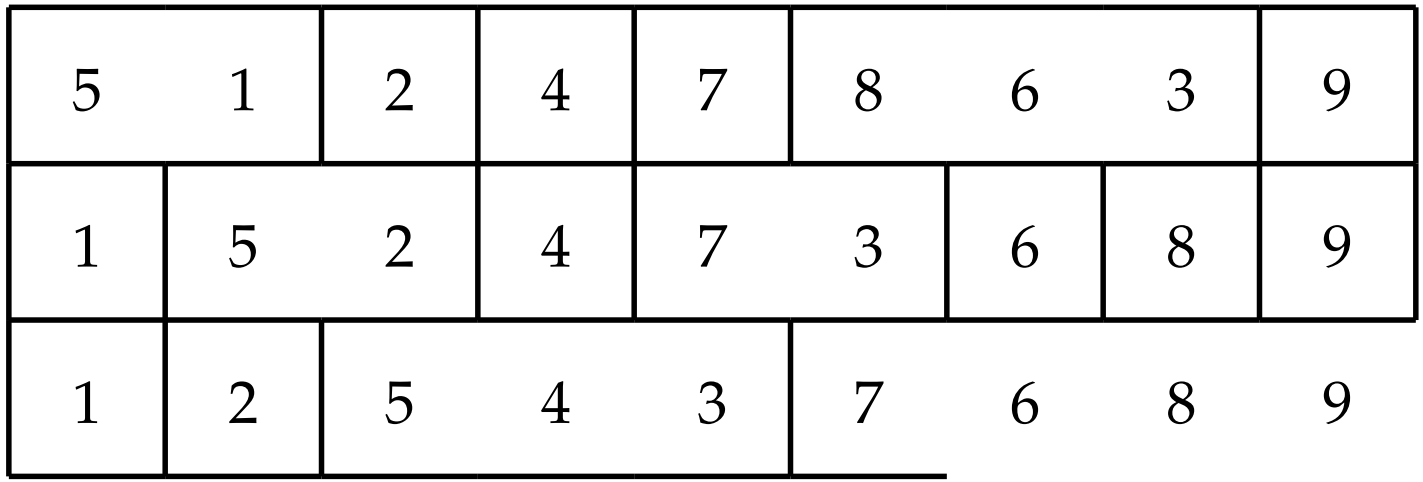
5	1	2	4	7	8	6	3	9
1	5	2	4	7	3	6	8	9
1	2	5	4	3	7	6	8	9

1 2

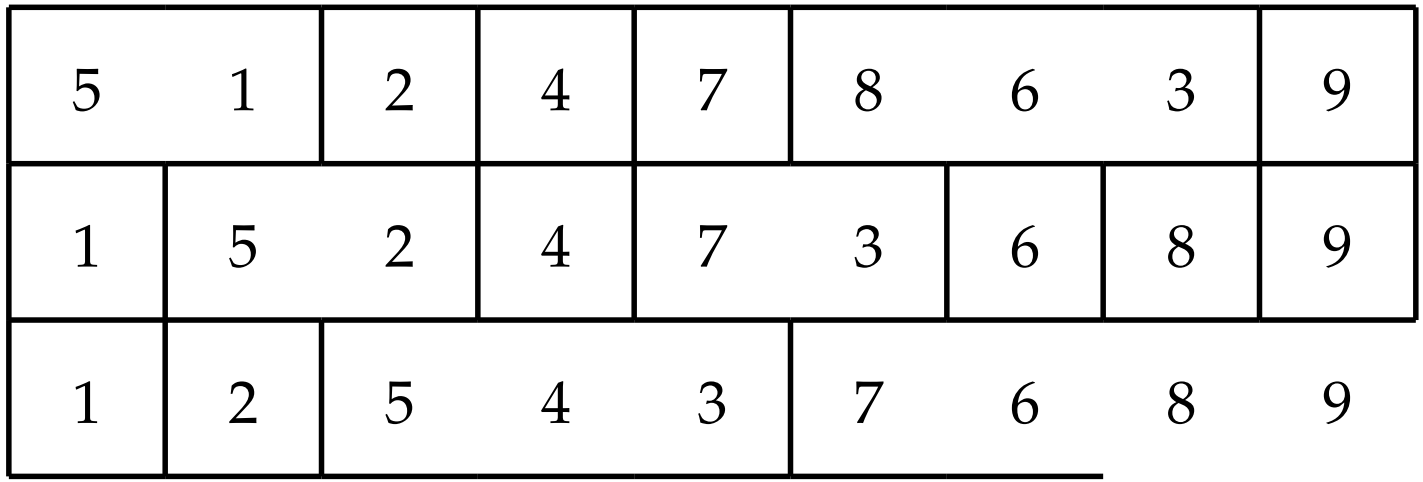


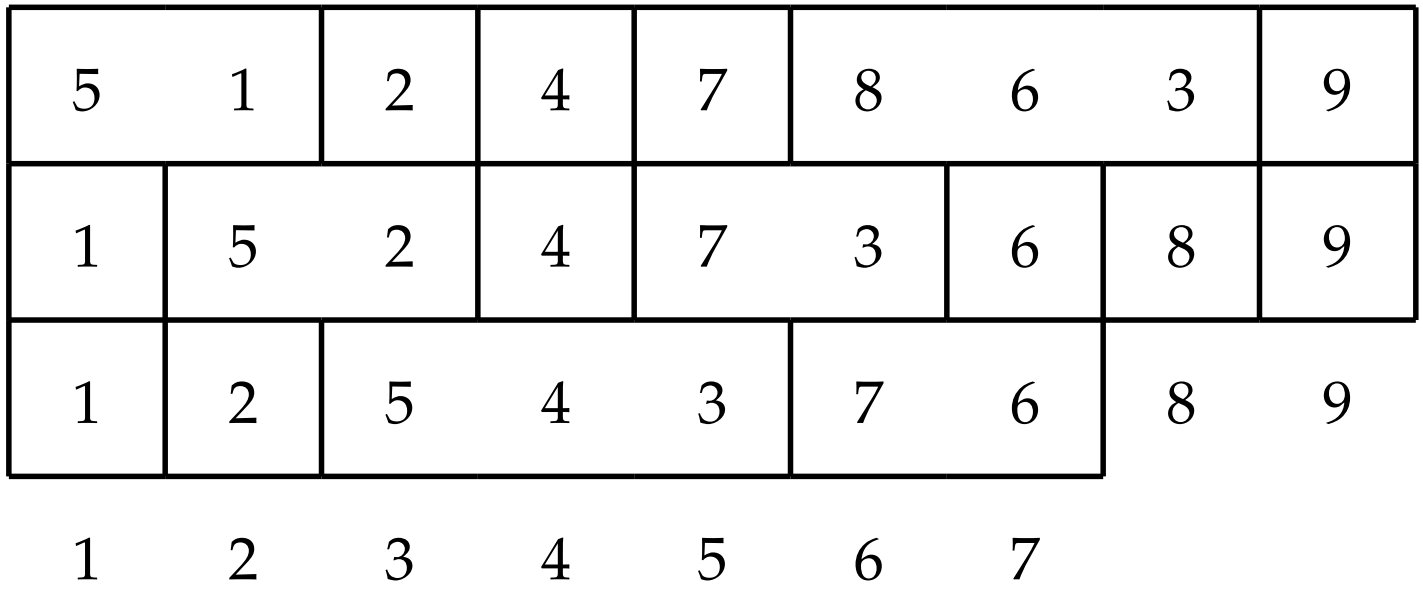
1 2

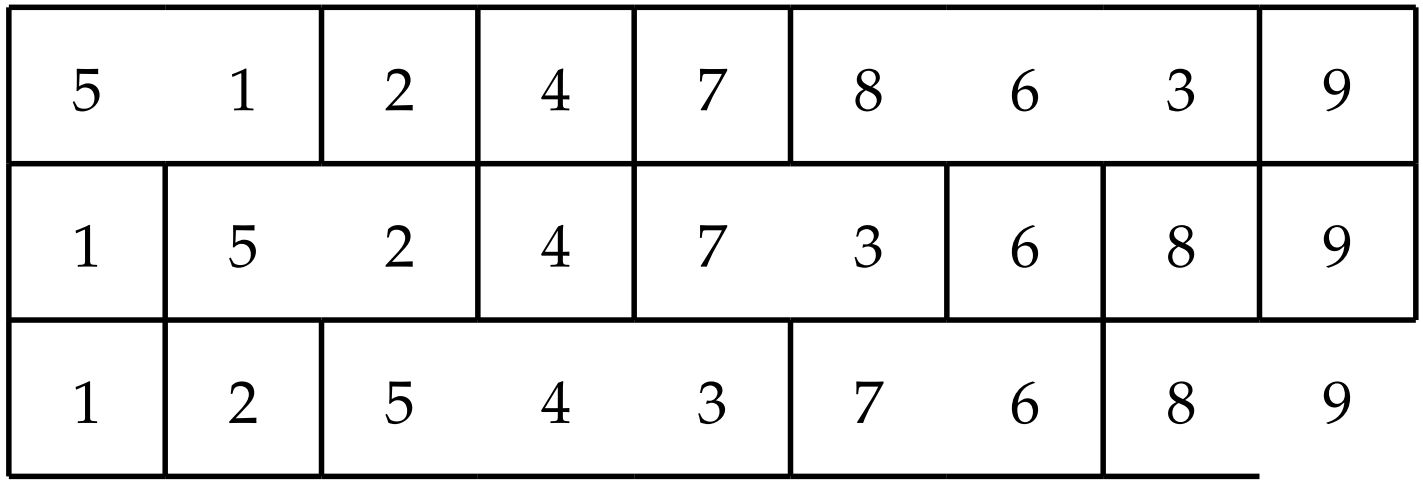

5 1 2 4 7 8 6 3 9
1 5 2 4 7 3 6 8 9
1 2 5 4 3 7 6 8 9
1 2 3 4 5

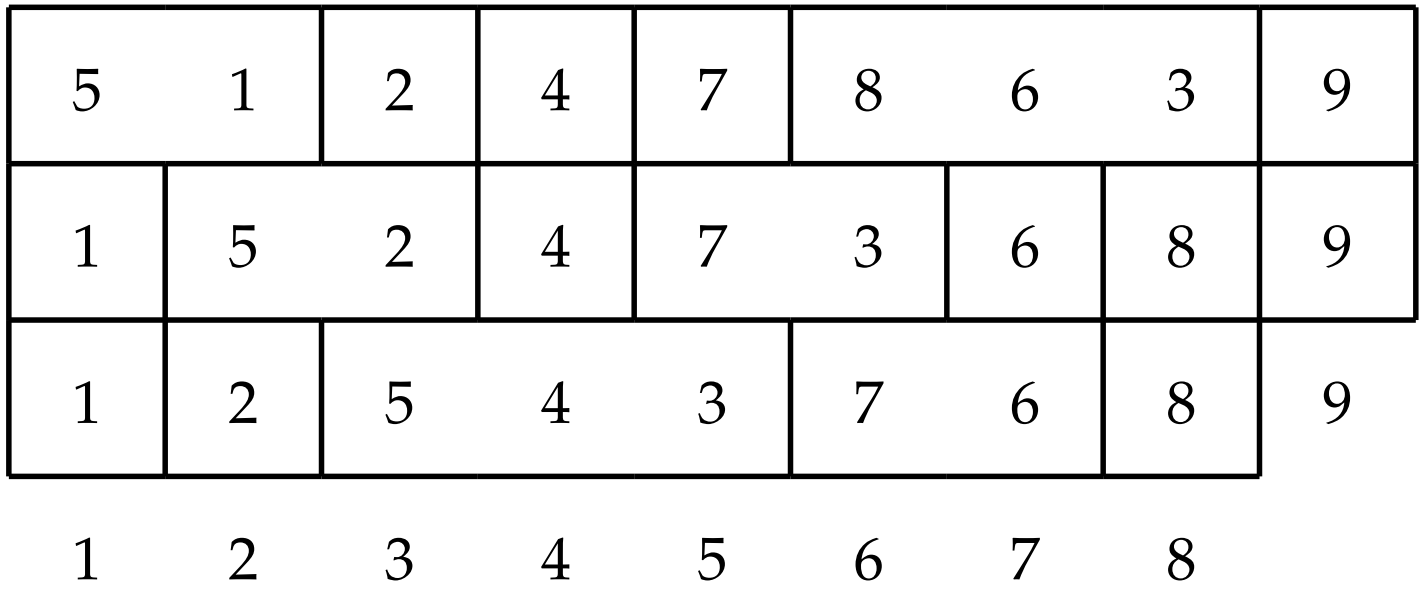


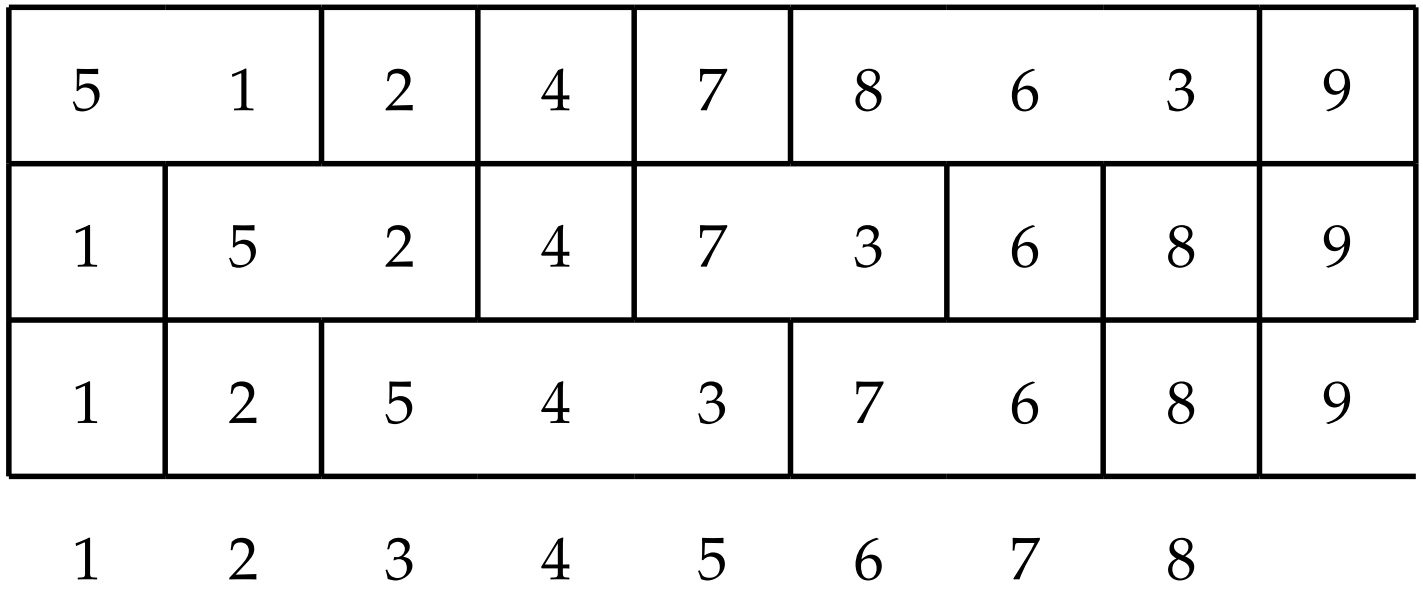
1 2 3 4 5

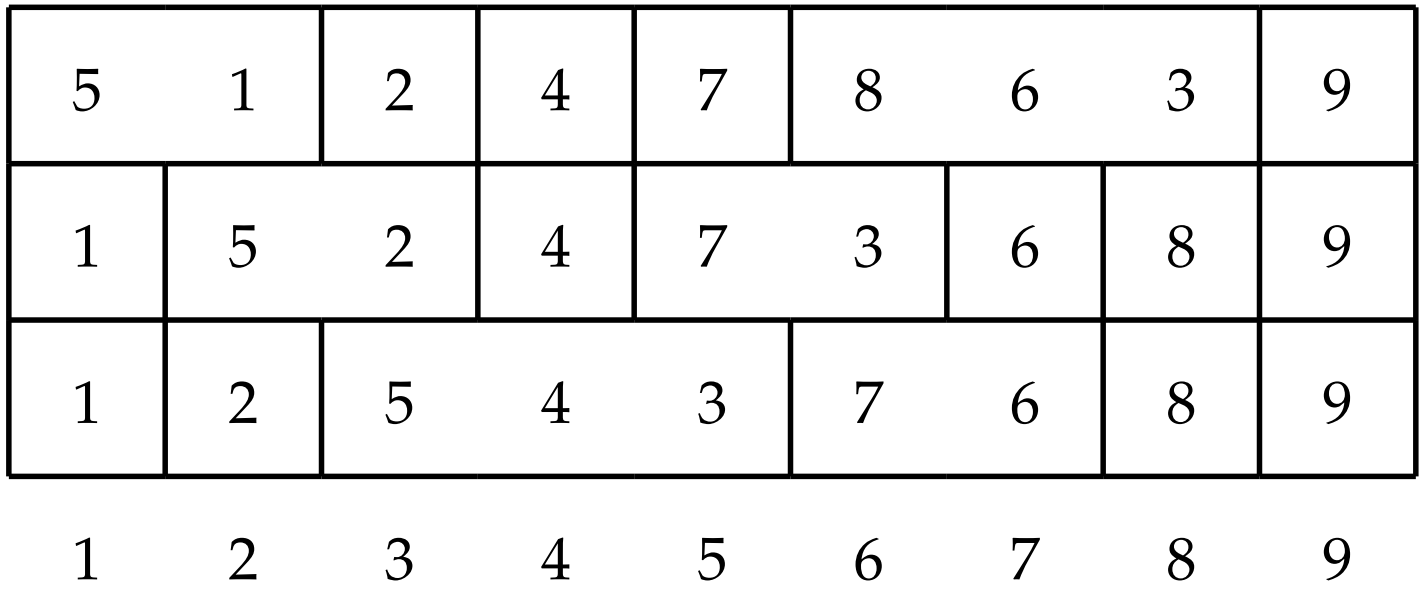

5 1 2 4 7 8 6 3 9
1 5 2 4 7 3 6 8 9
1 2 5 4 3 7 6 8 9
1 2 3 4 5

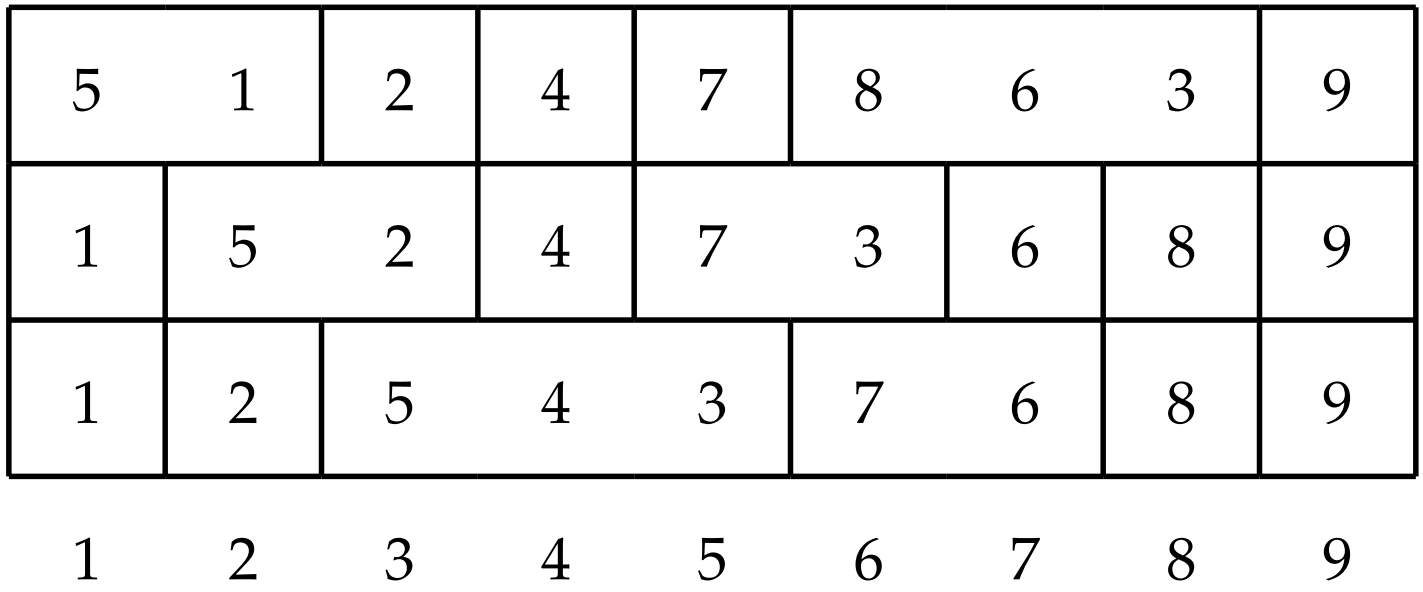

5 1 2 4 7 8 6 3 9
1 5 2 4 7 3 6 8 9
1 2 5 4 3 7 6 8 9
1 2 3 4 5 6 7


5 1 2 4 7 8 6 3 9
1 5 2 4 7 3 6 8 9
1 2 5 4 3 7 6 8 9
1 2 3 4 5 6 7


5 1 2 4 7 8 6 3 9
1 5 2 4 7 3 6 8 9
1 2 5 4 3 7 6 8 9
1 2 3 4 5 6 7 8


5 1 2 4 7 8 6 3 9
1 5 2 4 7 3 6 8 9
1 2 5 4 3 7 6 8 9
1 2 3 4 5 6 7 8

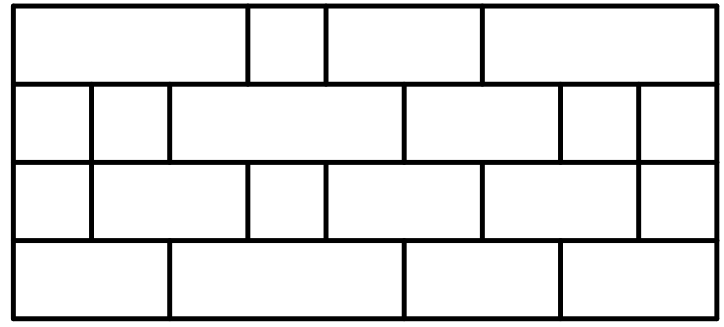

5 1 2 4 7 8 6 3 9
1 5 2 4 7 3 6 8 9
1 2 5 4 3 7 6 8 9
1 2 3 4 5 6 7 8 9



- A sorting trace of length 9 and order 3.
- Numbers within a block are in decreasing order.
- Adjacent numbers in different blocks form an ascent.
- Each perm is the "blockwise reversal" of the one above.
- The last permutation is the identity.

7	5	2	4	9	1	8	6	3
2	5	7	4	1	9	3	6	8
2	5	1	4	7	3	9	6	8
2	1	5	4	3	7	6	9	8
1	2	3	4	5	6	7	8	9

A sorting trace



Its sorting plan

a d d a a d a d d a
a a a d d a d a a a
a a d a a d a d a a
a d a d d a d a d a

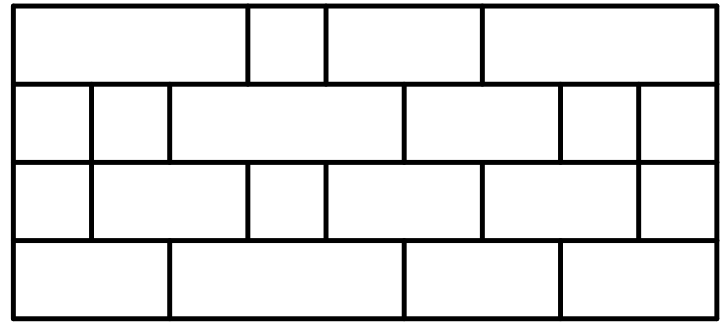
The same sorting plan

0, 9, 10, 5, 5, 10, 5, 10, 9, 0

. . . and its encoding

7	5	2	4	9	1	8	6	3
2	5	7	4	1	9	3	6	8
2	5	1	4	7	3	9	6	8
2	1	5	4	3	7	6	9	8
1	2	3	4	5	6	7	8	9

A sorting trace



Its sorting plan

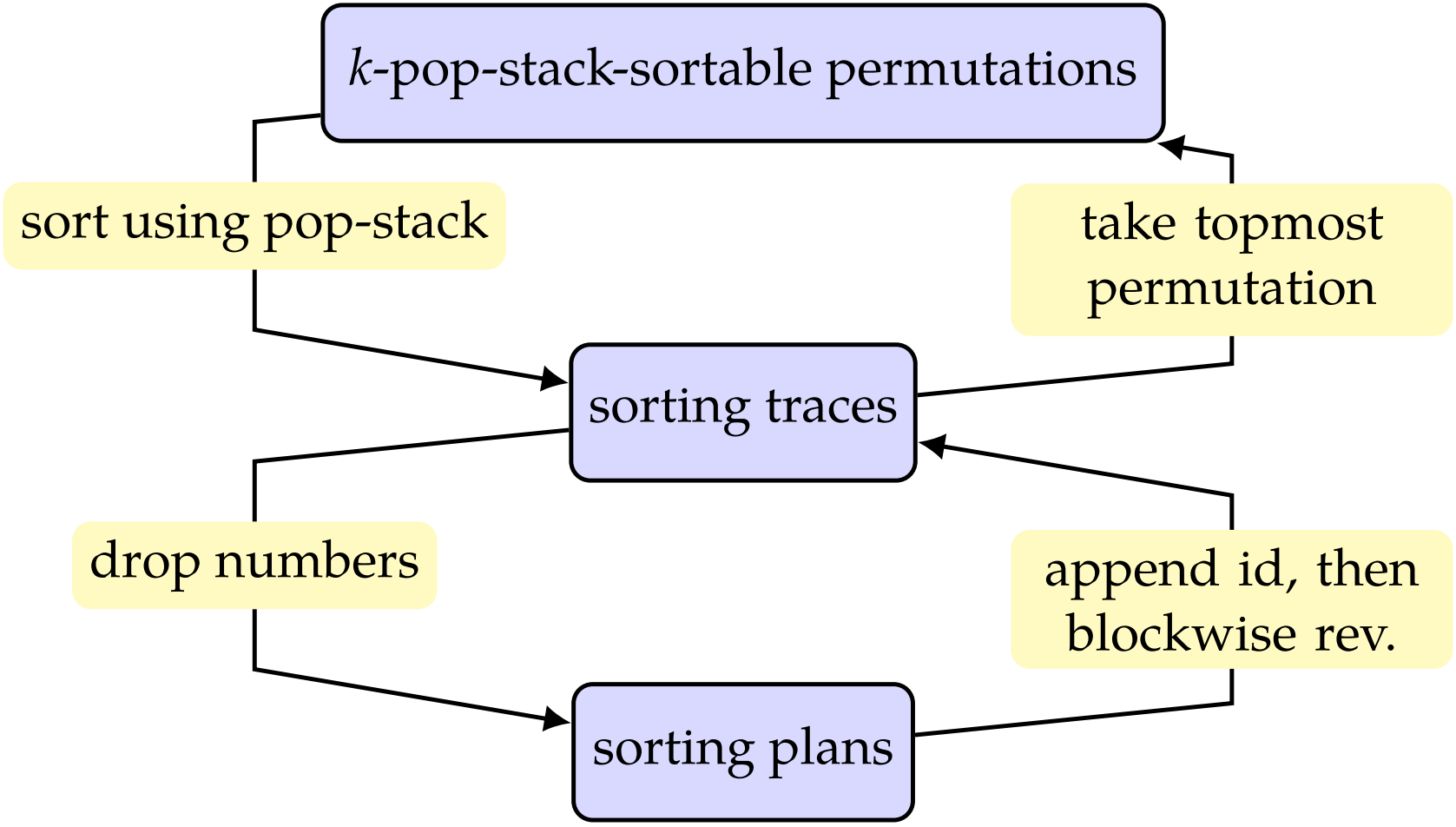
0 1 1 0 0 1 0 1 1 0
0 0 0 1 1 0 1 0 0 0
0 0 1 0 0 1 0 1 0 0
0 1 0 1 1 0 1 0 1 0

The same sorting plan

$0, 9, 10, 5, 5, 10, 5, 10, 9, 0$

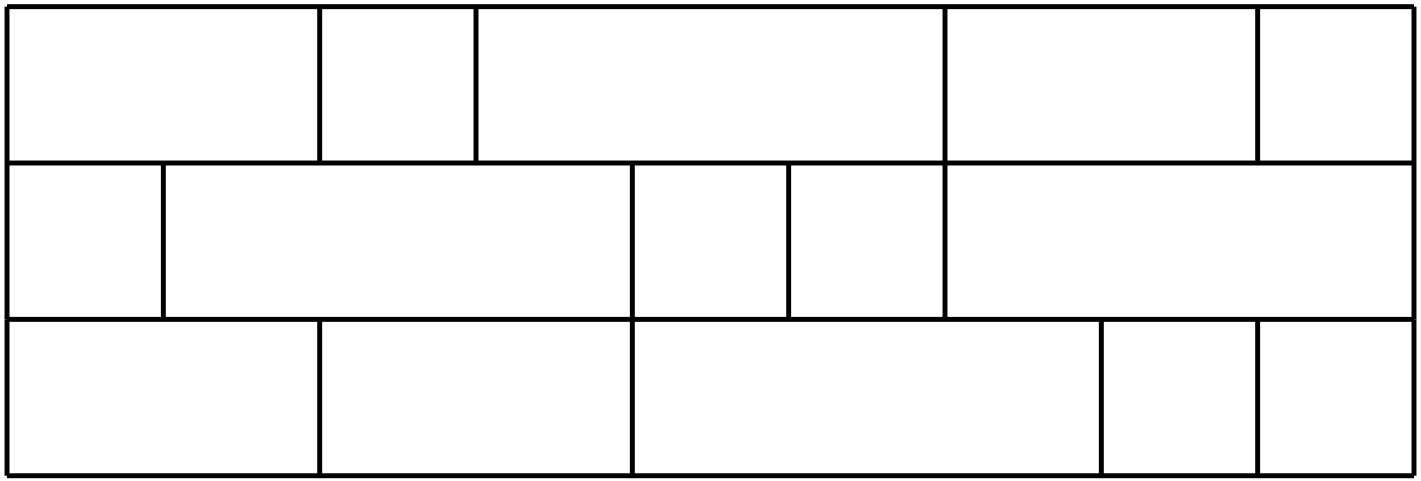
. . . and its encoding

Basic bijections

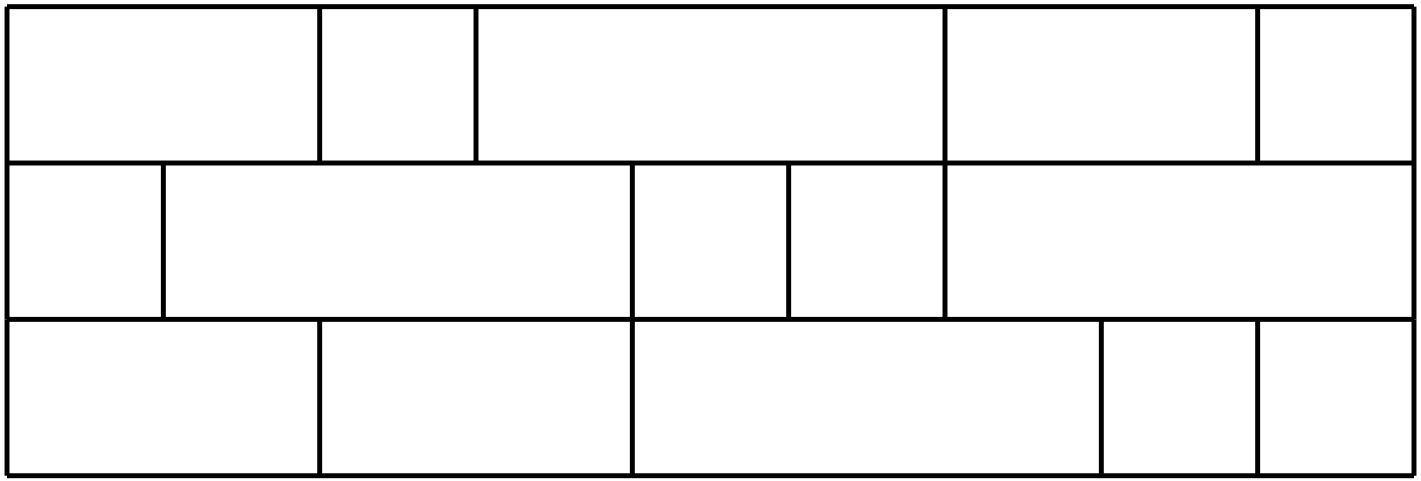


Bijection between k-pop-stack-sortable permutations of $[n]$ and sorting plans of length n and order k

An operation array of length 9 and order 3:

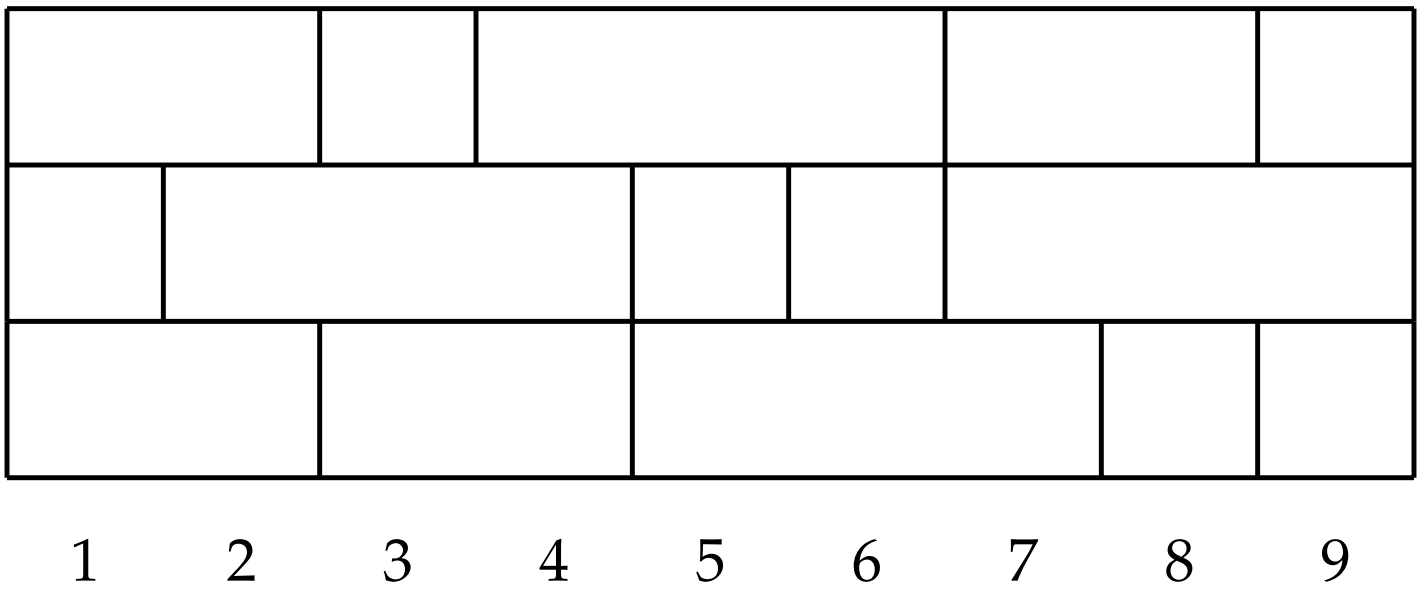


An operation array of length 9 and order 3:



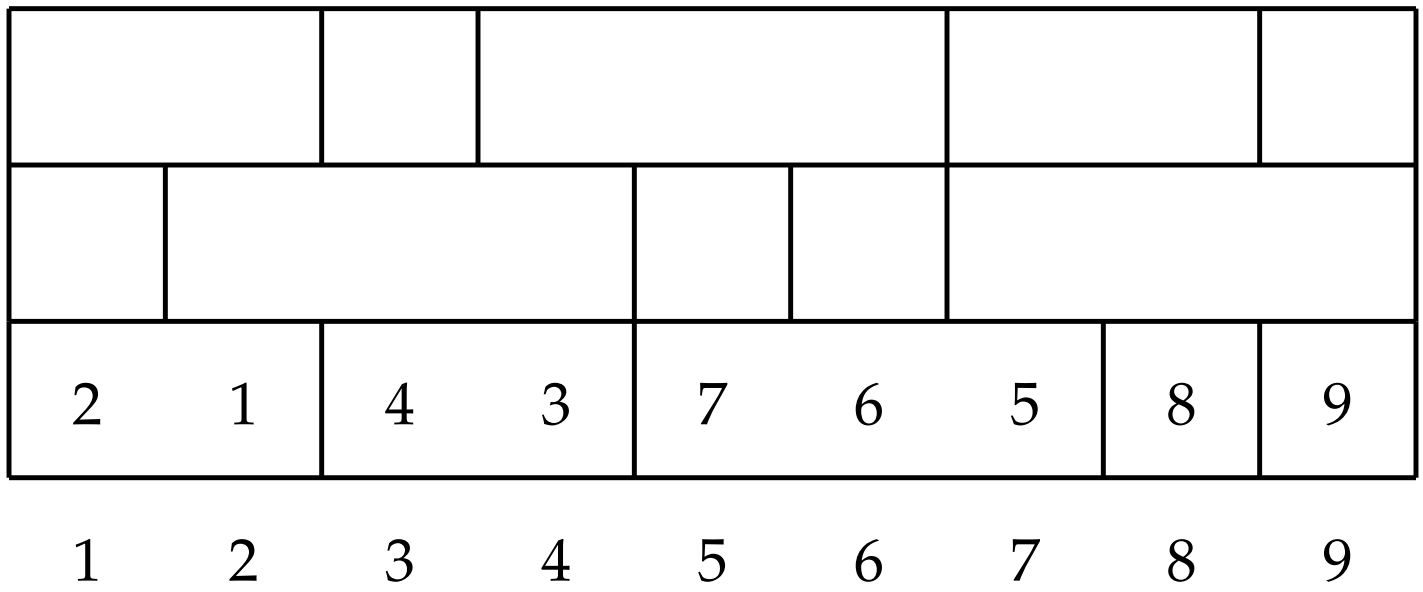
- Assume there is a sorting trace with this operation array.

An operation array of length 9 and order 3:



- Assume there is a sorting trace with this operation array.
- The last permutation must be the identity.

An operation array of length 9 and order 3:



- Assume there is a sorting trace with this operation array.
- The last permutation must be the identity.
- Each perm is the "blockwise reversal" of the perm above.

An operation array of length 9 and order 3:

2	3	4	1	7	6	9	8	5
2	1	4	3	7	6	5	8	9
1	2	3	4	5	6	7	8	9

- Assume there is a sorting trace with this operation array.
- The last permutation must be the identity.
- Each perm is the "blockwise reversal" of the perm above.

An operation array of length 9 and order 3:

3	2	4	6	7	1	8	9	5
2	3	4	1	7	6	9	8	5
2	1	4	3	7	6	5	8	9
1	2	3	4	5	6	7	8	9

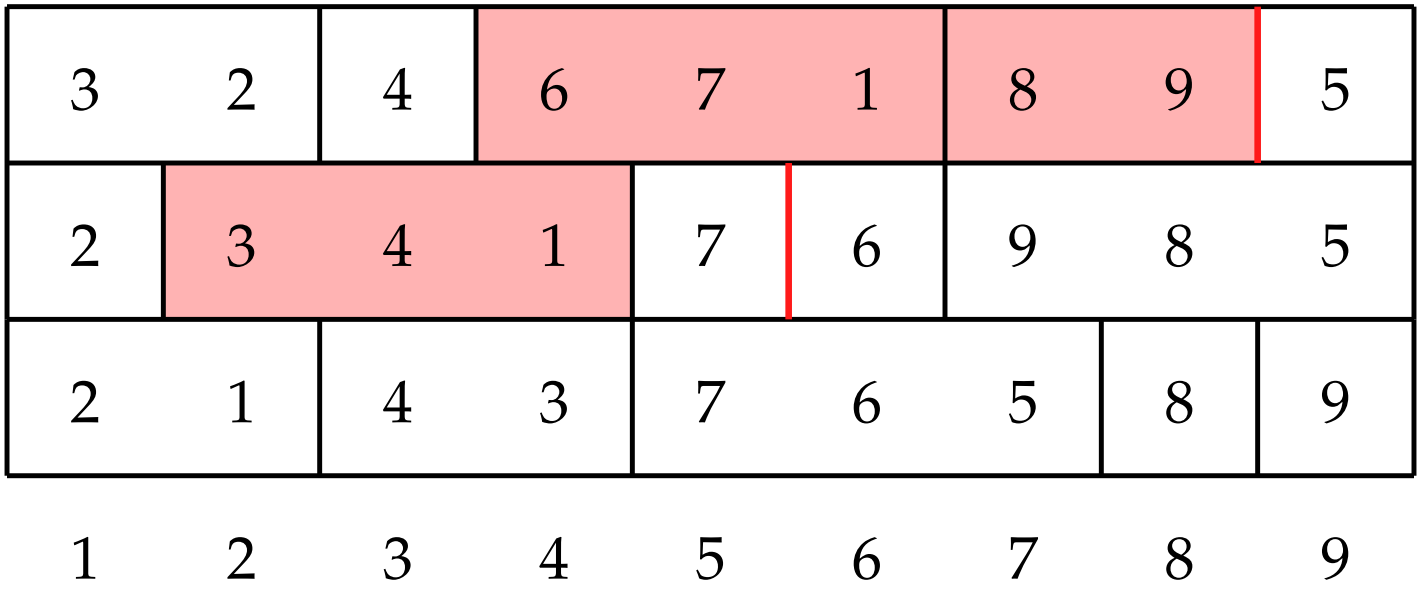
- Assume there is a sorting trace with this operation array.
- The last permutation must be the identity.
- Each perm is the "blockwise reversal" of the perm above.

An operation array of length 9 and order 3:

3	2	4	6	7	1	8	9	5
2	3	4	1	7	6	9	8	5
2	1	4	3	7	6	5	8	9
1	2	3	4	5	6	7	8	9

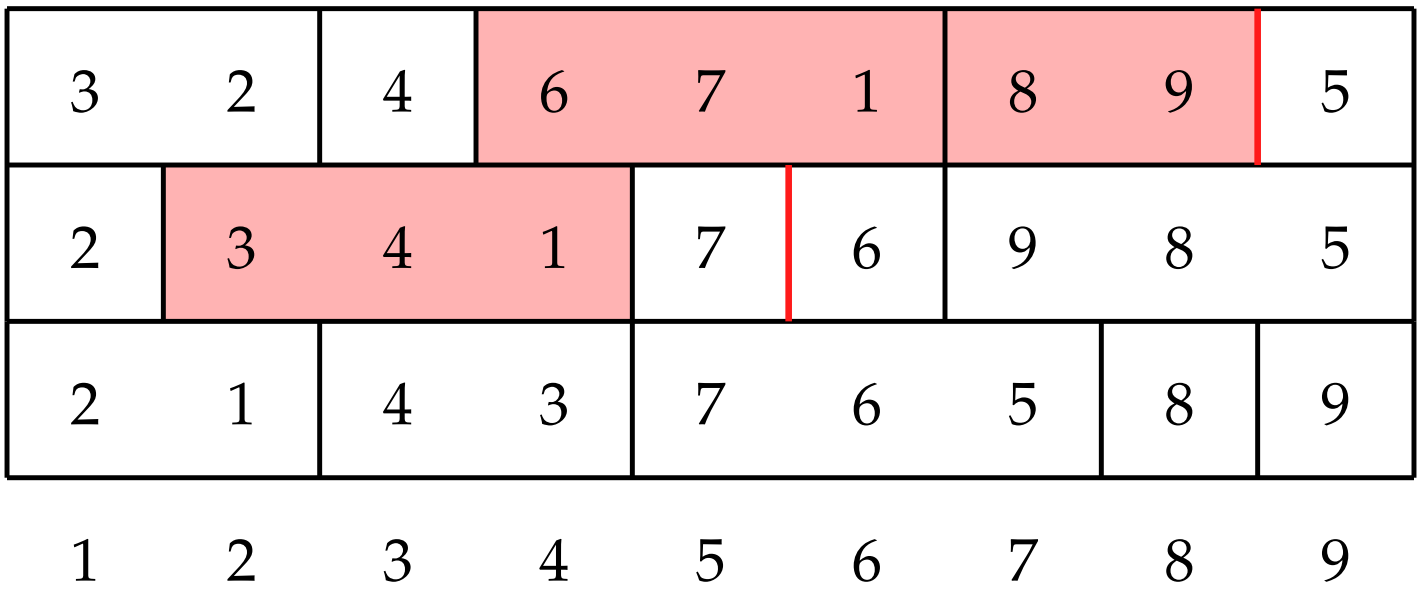
- Assume there is a sorting trace with this operation array.
- The last permutation must be the identity.
- Each perm is the "blockwise reversal" of the perm above.

An operation array of length 9 and order 3:

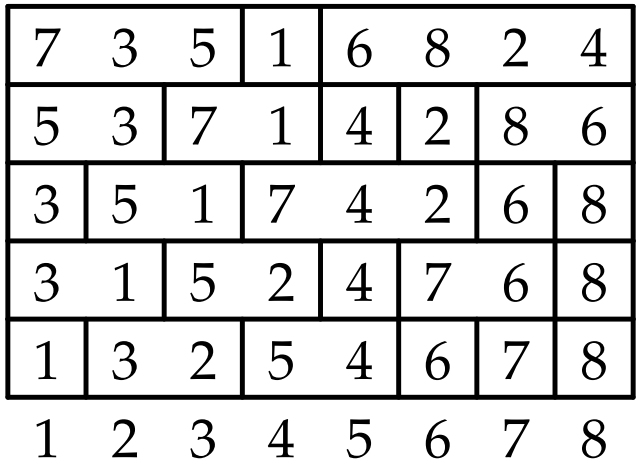


- Assume there is a sorting trace with this operation array.
- The last permutation must be the identity.
- Each perm is the "blockwise reversal" of the perm above.

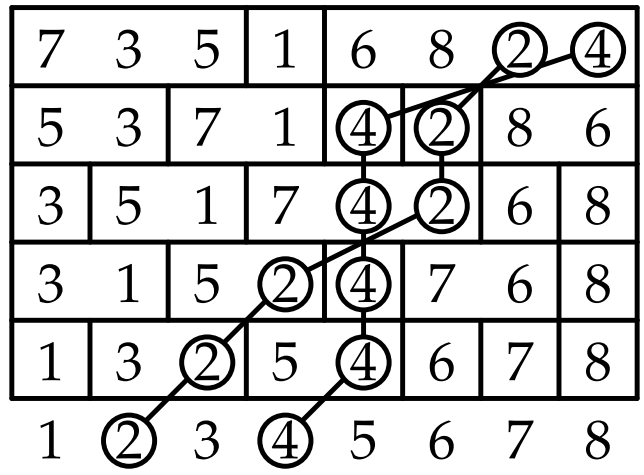
An operation array of length 9 and order 3:



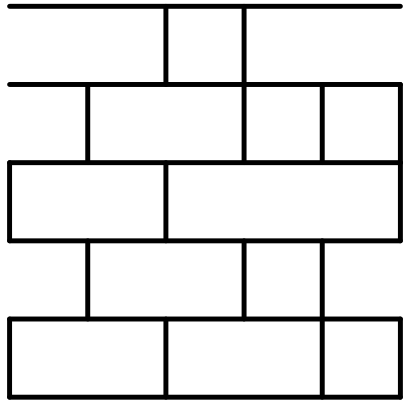
- Assume there is a sorting trace with this operation array.
- The last permutation must be the identity.
- Each perm is the "blockwise reversal" of the perm above.
- This semitrace is not a (proper) trace, so the operation array is *not* a sorting plan!



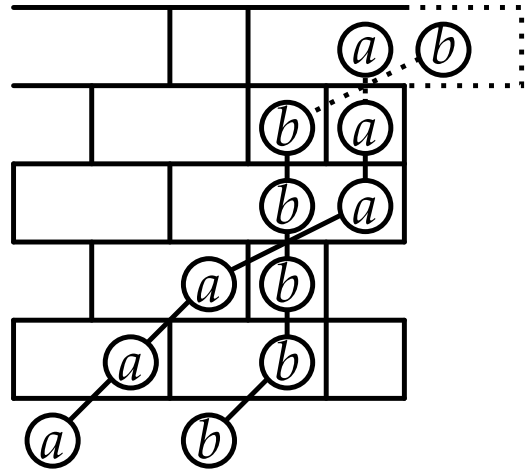
An example semitrace



The progress of 2 and 4



A forbidden segment

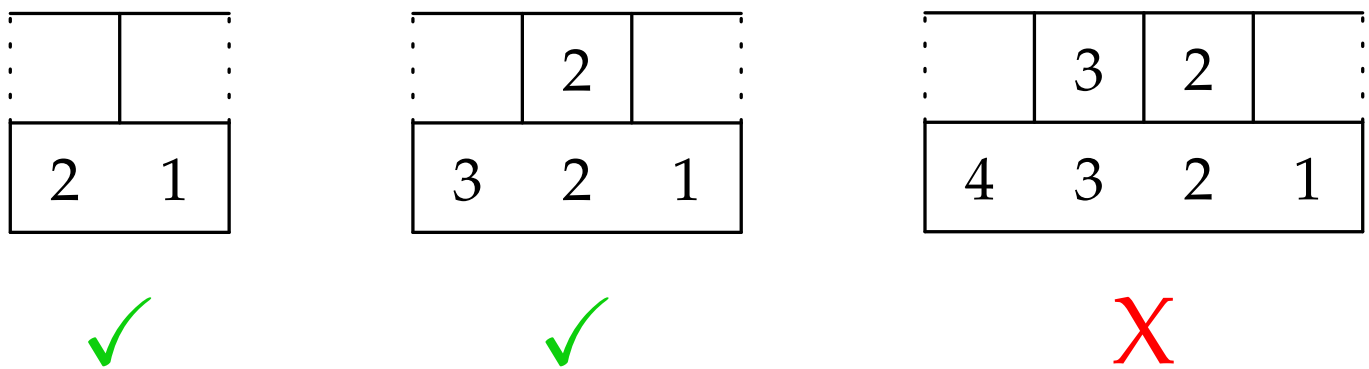


Following a and b

Lemma

In a sorting plan, each but the first row has blocks of size ≤ 3.

Proof

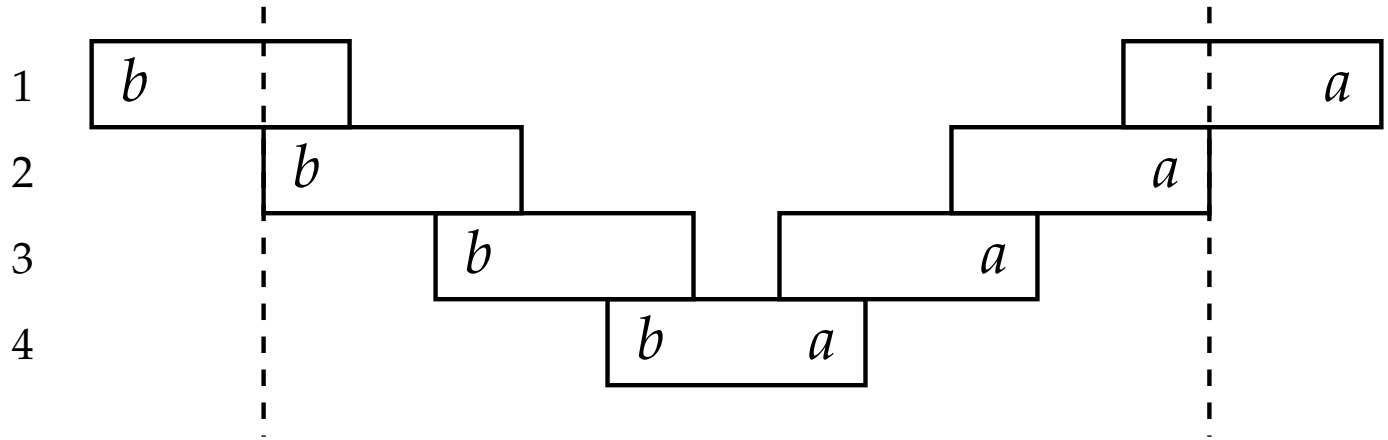


✓ ✓ X

Lemma

If T is a forbidden segment in some semitrace of order k, then $|T| \leq 4k - 5$.

Proof

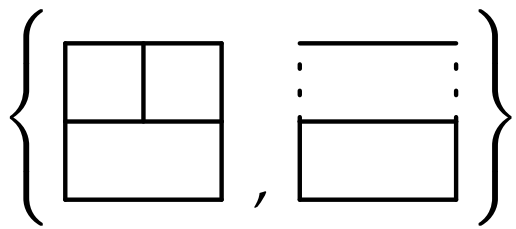


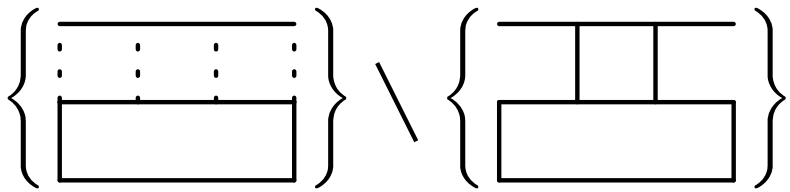
Corollary

For a fixed k, there are finitely many forbidden segments of order k, and they can be listed.

- For $k = 1$ there are no forbidden segments.
- For $k = 2$ the forbidden segments are

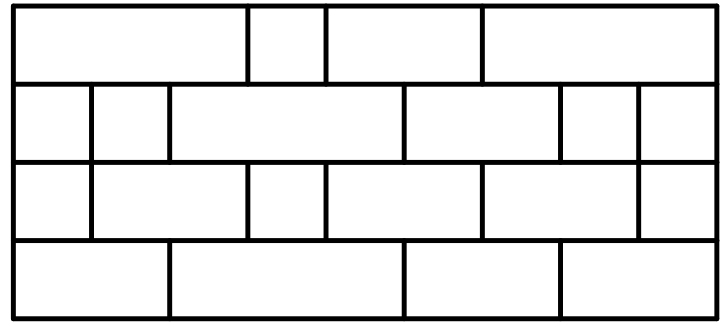
1 $\emptyset$

2 

3 

7	5	2	4	9	1	8	6	3
2	5	7	4	1	9	3	6	8
2	5	1	4	7	3	9	6	8
2	1	5	4	3	7	6	9	8
1	2	3	4	5	6	7	8	9

A sorting trace



Its sorting plan

a d d a a d a d d a
a a a d d a d a a a
a a d a a d a d a a
a d a d d a d a d a

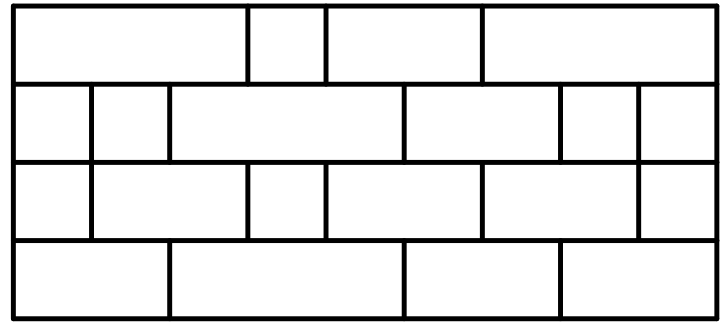
The same sorting plan

$0, 9, 10, 5, 5, 10, 5, 10, 9, 0$

. . . and its encoding

7	5	2	4	9	1	8	6	3
2	5	7	4	1	9	3	6	8
2	5	1	4	7	3	9	6	8
2	1	5	4	3	7	6	9	8
1	2	3	4	5	6	7	8	9

A sorting trace



Its sorting plan

0 1 1 0 0 1 0 1 1 0
0 0 0 1 1 0 1 0 0 0
0 0 1 0 0 1 0 1 0 0
0 1 0 1 1 0 1 0 1 0

The same sorting plan

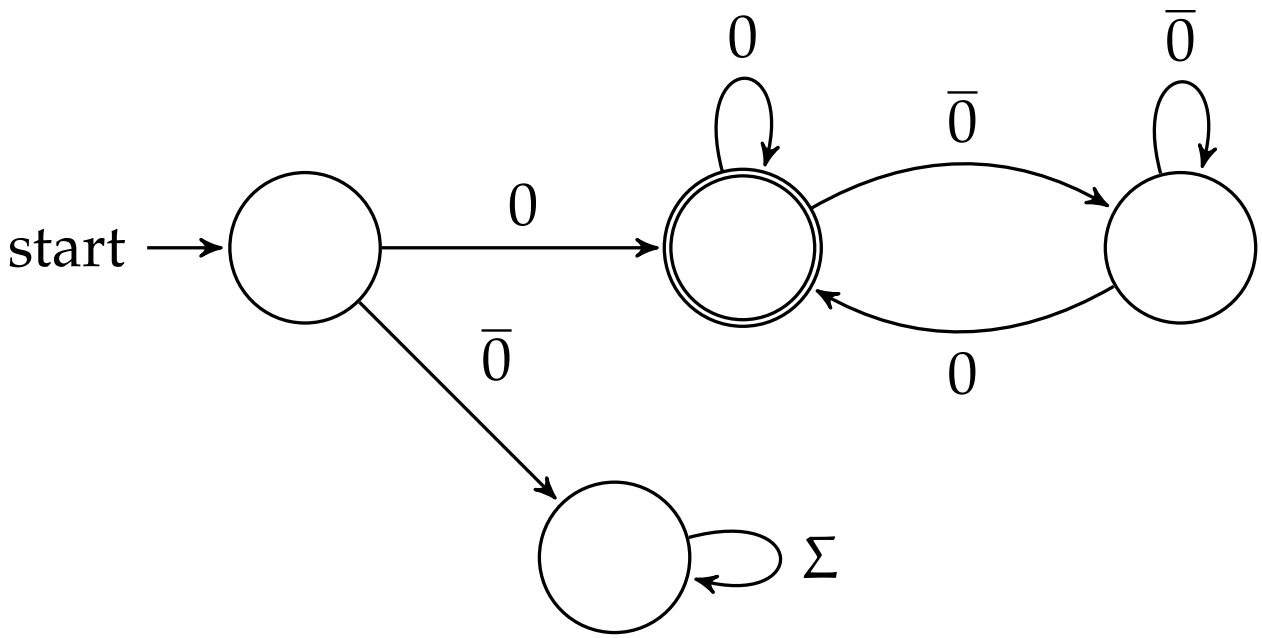
$0, 9, 10, 5, 5, 10, 5, 10, 9, 0$

. . . and its encoding

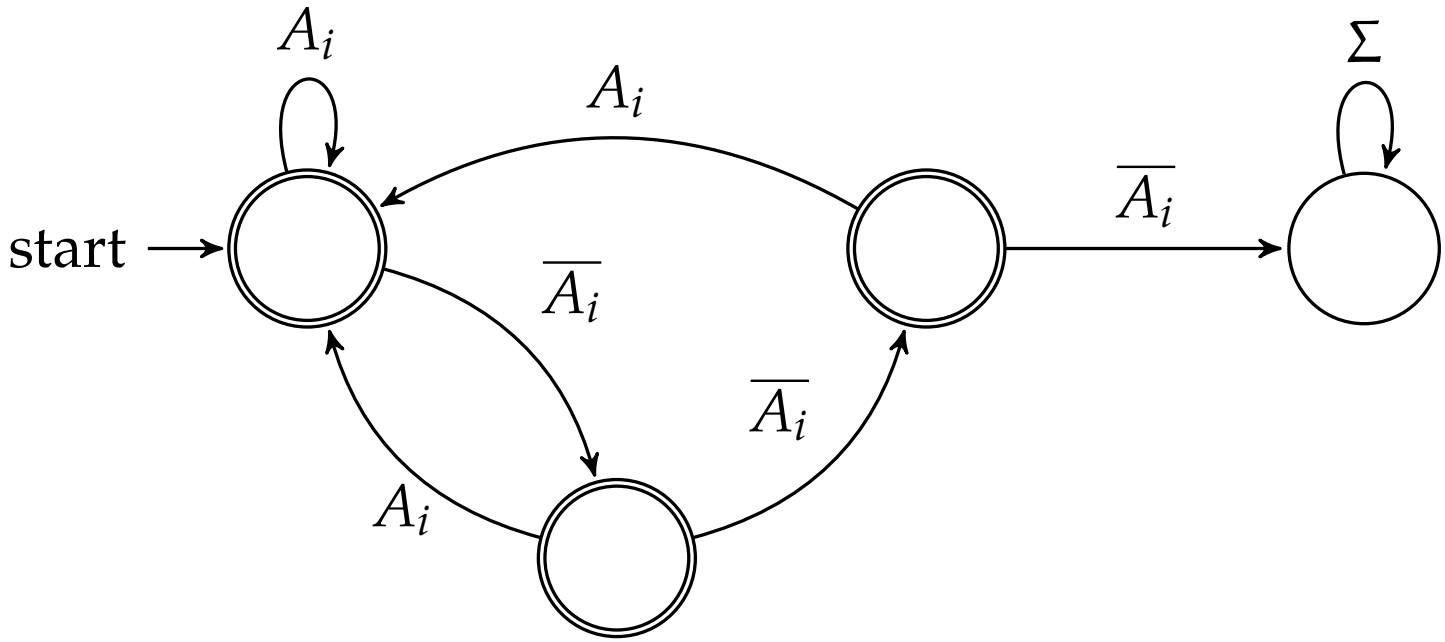
- Encode an operation array of length n and order k as a sequence of n integers, each in the range

$$\Gamma = \{0, 1, \ldots, 2^k - 1\}.$$

- Thus operation arrays can be seen as strings in Γ^*.
- A DFA for the language of operation arrays, W, is

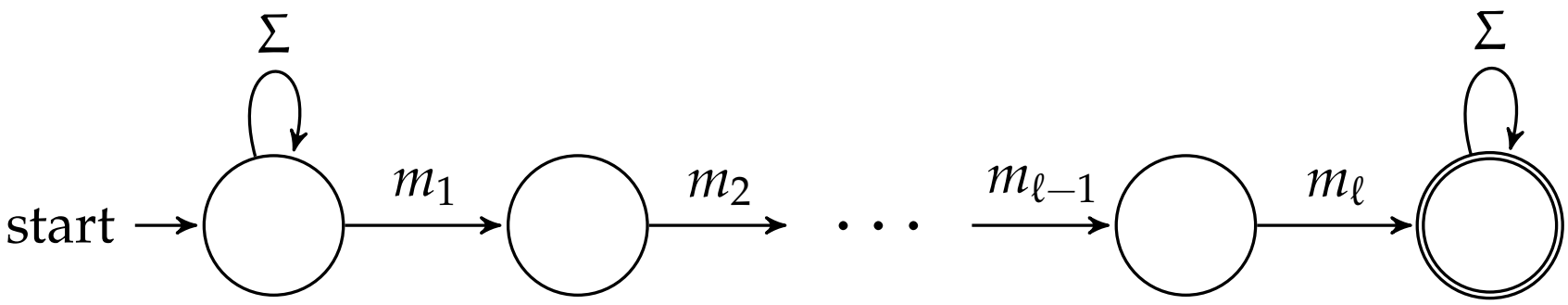


- Let A_i be the set of symbols in Σ that represent a column that has a bar in the ith row
- The following DFA, R_i, recognizes the operation arrays that have blocks of size at most 3 in row i:



- Operation arrays that have blocks of size at most 3 in all but the first row is thus recognized by the DFA $W \cap R_2 \cap \cdots \cap R_k$.

- Recall that sorting plans are characterized by avoiding forbidden segments.
- Let us encode a segment M in the same manner as we encode operation arrays, resulting in the sequence $m_1, \ldots, m_\ell$.
- An operation array A contains the segment M if and only if the encoding of A contains $m_1 \cdots m_\ell$ as a factor.
- The following NFA, Q_M, recognizes strings over Σ that contain the encoding of M as a factor:



Let $\mathcal{F}$ be the set of forbidden segments. Then

$$S \;=\; W \cap \bigcap_{i=2}^{k} R_i \cap \bigcap_{F \in \mathcal{F}} \overline{Q_F}$$

recognizes the set of sorting plans.

Theorem

The language $S = \{\, w \in \Sigma^* : w$ is a sorting plan $\}$ is regular.

Main result

- Let $p_k(n)$ be the number of k-pop-stack-sortable permutations of $[n]$.
- Let $P_k(x) = \sum_{n \geq 0} p_k(n) x^n$.

Theorem

The generating function $P_k(x)$ is rational.

Data

This has been implemented. Running on a big cluster we got:

k	1	2	3	4	5	6
degree	1	3	10	25	71	213
growth rate	2.0000	2.6590	3.4465	4.2706	5.1166	5.9669
vertices	4	5	12	32	99	339
edges	8	11	34	120	477	2010

All the generating functions, source code, and text files defining the DFAs can be found on GitHub:

`github.com/SuprDewd/popstacks`

Generating functions

1 $$(x-1)\,/\,(2x-1)$$

2 $$(x^3+x^2+x-1)\,/\,(2x^3+x^2+2x-1)$$

3 $$(2x^{10}+4x^9+2x^8+5x^7+11x^6+8x^5+6x^4+6x^3+2x^2+x-1)\,/\,(4x^{10}+8x^9+4x^8+10x^7+22x^6+16x^5+8x^4+6x^3+2x^2+2x-1)$$

4 $$(64x^{25}+448x^{24}+1184x^{23}+1784x^{22}+2028x^{21}+1948x^{20}+1080x^{19}+104x^{18}-180x^{17}+540x^{16}+1156x^{15}+696x^{14}+252x^{13}+238x^{12}+188x^{11}+502x^{10}+806x^9+544x^8+263x^7+185x^6+99x^5+33x^4+13x^3+3x^2+x-1)\,/\,(128x^{25}+896x^{24}+2368x^{23}+3568x^{22}+3928x^{21}+3064x^{20}+176x^{19}-2304x^{18}-2664x^{17}-1580x^{16}-352x^{15}-576x^{14}-1104x^{13}-760x^{12}-138x^{11}+686x^{10}+1238x^9+869x^8+382x^7+210x^6+102x^5+27x^4+12x^3+3x^2+2x-1)$$

5, 6 too large to display