Homomorphisms on Noncommutative Symmetric Functions and Permutation Enumeration

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Partially based on joint work with Ira M. Gessel.

Outline

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Permutations and Descents

- A permutation of length n (or, n-permutation) $\pi = \pi_1\pi_2\cdots\pi_n$ is a linear ordering of $[n] := \{1, 2, \ldots, n\}$.

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- The set of n-permutations is denoted $\mathfrak{S}_n$.
- We say that $k \in [n-1]$ is a descent of $\pi \in \mathfrak{S}_n$ if $\pi_k > \pi_{k+1}$. The descent number $\operatorname{des}(\pi)$ is the number of descents of π.

Increasing Runs and Descent Compositions

- Descents separate permutations into increasing runs: maximal increasing consecutive subsequences.
- Call the tuple of increasing run lengths of π the descent composition of π, denoted $\mathrm{Comp}(\pi)$.

Example

Given $\pi = 172346589$, we have $\mathrm{Comp}(\pi) = (2, 4, 3)$.

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- The notions of descents and increasing runs extend to words on any totally ordered alphabet (such as the positive integers $\mathbb{P}$).

Example

The increasing runs of the word $w = 11526249$ are 115, 26, and 249, so $\mathrm{Comp}(w) = (3, 2, 3)$ and $\mathrm{des}(w) = 2$.

Ribbon Functions

- Let $X_1, X_2, \ldots$ be noncommuting variables. Given a composition $L = (L_1, L_2, \ldots, L_k)$, define the ribbon function $\mathbf{r}_L$ by

$$\mathbf{r}_L := \sum_{(i_1,\ldots,i_n)} X_{i_1} X_{i_2} \cdots X_{i_n}$$

over all $(i_1, \ldots, i_n)$ satisfying

$$\underbrace{i_1 \leq \cdots \leq i_{L_1}}_{L_1} > \underbrace{i_{L_1+1} \leq \cdots \leq i_{L_1+L_2}}_{L_2} > \cdots > \underbrace{i_{L_1+\cdots+L_{k-1}+1} \leq \cdots \leq i_n}_{L_k}.$$

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Example

The words 221552 and 374443 have descent composition $(2, 3, 1)$, so $X_2^2 X_1 X_5^2 X_2$ and $X_3 X_7 X_4^3 X_3$ are both terms in $\mathbf{r}_{(2,3,1)}$.

Noncommutative Symmetric Functions

- Let $\mathbf{Sym}_n$ be the vector space with basis $\{\mathbf{r}_L\}_{L\vDash n}$. Then

$$\mathbf{Sym} := \bigoplus_{n=0}^{\infty} \mathbf{Sym}_n$$

is a subalgebra of $\mathbb{Q}\langle\langle X_1, X_2, \ldots \rangle\rangle$ called the algebra of noncommutative symmetric functions.

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- Let $\mathbf{h}_n := \mathbf{r}_{(n)} = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} X_{i_1} X_{i_2} \cdots X_{i_n}$. Then the $\mathbf{h}_n$ are algebraically independent and generate $\mathbf{Sym}$.

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Our Approach

- Many permutation enumeration formulas involving runs or descents can be proven in the following way:
  1. Derive a lifting of the formula in **Sym**.
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  1. Derive a lifting of the formula in **Sym**.
  2. Apply an appropriate homomorphism.
- The simplest such homomorphism is $\Phi\colon \mathbf{Sym} \to \mathbb{Q}[[x]]$ defined by

$$\Phi(\mathbf{h}_n) = \frac{x^n}{n!}.$$

Then

$$\Phi(\mathbf{r}_L) = \beta(L)\frac{x^n}{n!}$$

where $\beta(L)$ is the number of permutations with descent composition $L \vDash n$.

David and Barton's Formula

Lemma (Gessel & Z. 2014)

$$\sum_{\substack{L \\ \text{all parts}<m}} \mathbf{r}_L = \left[\sum_{n=0}^{\infty}(\mathbf{h}_{mn} - \mathbf{h}_{mn+1})\right]^{-1}$$

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- By applying the homomorphism Φ, we get:

Theorem (David & Barton 1962)

Let $a_{n,m}$ be the number of n-permutations with every increasing run having length less than m (i.e., avoiding the consecutive pattern $\underline{12\cdots m}$). Then

$$\sum_{n=0}^{\infty} a_{n,m}\frac{x^n}{n!} = \left[\sum_{n=0}^{\infty}\left(\frac{x^{mn}}{(mn)!} - \frac{x^{mn+1}}{(mn+1)!}\right)\right]^{-1}.$$

Alternating Analogues

- We say that $k \in [n-1]$ is an alternating descent of $\pi \in \mathfrak{S}_n$ if k is an odd descent or an even ascent.

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Example

The alternating descents of $\pi = 17645823$ are 3 and 4, and the alternating runs of π are 176, 4, and 5823.

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- The alternating descent number and alternating descent composition are defined analogously.

An Alternating Analogue of Φ

- The nth Euler number E_n is defined by

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- Define $\hat{\Phi}\colon \mathbf{Sym} \to \mathbb{Q}[[x]]$ by

$$\hat{\Phi}(\mathbf{h}_n) = E_n \frac{x^n}{n!}.$$

Then

$$\hat{\Phi}(\mathbf{r}_L) = \hat{\beta}(L) \frac{x^n}{n!}$$

where $\hat{\beta}(L)$ is the number of permutations with alternating descent composition $L \vDash n$.

An Alternating Analogue of David and Barton's Formula

- Applying $\hat{\Phi}$ to $\displaystyle\sum_{\substack{L \\ \text{all parts}<m}} \mathbf{r}_L = \left[\sum_{n=0}^{\infty}(\mathbf{h}_{mn} - \mathbf{h}_{mn+1})\right]^{-1}$ yields:

Theorem (Gessel & Z. 2014)

Let $\hat{a}_{n,m}$ be the number of n-permutations with every alternating run having length less than m. Then

$$\sum_{n=0}^{\infty} \hat{a}_{n,m}\frac{x^n}{n!} = \left[\sum_{n=0}^{\infty}\left(E_{mn}\frac{x^{mn}}{(mn)!} - E_{mn+1}\frac{x^{mn+1}}{(mn+1)!}\right)\right]^{-1}.$$

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- The case $m = 3$ counts permutations with all peaks odd and all valleys even, answering a question of Liviu Nicolaescu.

A q-Analogue of Φ

- We say that $(i,j) \in [n]^2$ is an inversion of $\pi \in \mathfrak{S}_n$ if $i < j$ and $\pi_i > \pi_j$.
- The inversion number $\mathrm{inv}(\pi)$ is the number of inversions of π.

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- Define $\Phi_q \colon \mathbf{Sym} \to \mathbb{Q}[[q,x]]$ by

$$\Phi_q(\mathbf{h}_n) = \frac{x^n}{[n]_q!}.$$

Then

$$\Phi_q(\mathbf{r}_L) = \beta_q(L)\frac{x^n}{[n]_q!}$$

where

$$\beta_q(L) := \sum_{\pi \in \mathrm{Comp}(L)} q^{\mathrm{inv}(\pi)}.$$

A q-Analogue of David and Barton's Formula

- By applying the homomorphism Φ_q to

$$\sum_{\substack{L \\ \text{all parts} < m}} \mathbf{r}_L = \left[\sum_{n=0}^{\infty} (\mathbf{h}_{mn} - \mathbf{h}_{mn+1}) \right]^{-1},$$

we get:

Theorem (Elizalde 2016)

Let $a_{n,m}(q) = \sum_{\pi} q^{\mathrm{inv}(\pi)}$ over all n-permutations with every increasing run having length less than m (i.e., avoiding the consecutive pattern $\underline{12 \cdots m}$). Then

$$\sum_{n=0}^{\infty} a_{n,m}(q) \frac{x^n}{[n]_q!} = \left[\sum_{n=0}^{\infty} \left(\frac{x^{mn}}{[mn]_q!} - \frac{x^{mn+1}}{[mn+1]_q!} \right) \right]^{-1}.$$

Eulerian Polynomials

Lemma (Gessel 1977)

$$\sum_{L} t^{\mathrm{des}(L)+1}\mathbf{r}_L = (1-t)\Big(1 - t\sum_{n=0}^{\infty}(1-t)^n \mathbf{h}_n\Big)^{-1}$$

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- Applying Φ yields the well-known exponential generating function for the Eulerian polynomials $A_n(t)$ defined by

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- Applying Φ_q yields the q-exponential generating function for q-Eulerian polynomials (Stanley 1976).

The Run Theorem

- Gessel's run theorem (1977) can be used to derive noncommutative symmetric function formulas for counting permutations
  1. with restrictions on increasing runs;
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  1. with restrictions on increasing runs;
  2. by permutation statistics that are expressible in terms of increasing runs.

- See the following papers for more applications of this method to permutation enumeration:
  - I. M. Gessel and Y. Zhuang. Counting permutations by alternating descents. Electron. J. Combin., 21(4): Paper P4.23, 21 pp., 2014.
  - Y. Zhuang. Counting permutations by runs. J. Combin. Theory Ser. A, 142: 147–176, 2016.
  - Y. Zhuang. Eulerian polynomials and descent statistics. Adv. in Appl. Math., 90: 86–144, 2017.

  These include rederivations of known results by Carlitz, Elizalde and Noy, Entringer, Petersen, Remmel, Stanley, and Stembridge.

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- Given a permutation statistic st, define the inverse statistic ist by $\mathrm{ist}(\pi) = \mathrm{st}(\pi^{-1})$ for all π.

Example

Given $\pi = 24135$ we have $\pi^{-1} = 31425$, so $\mathrm{ides}(\pi) = 2$.

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Theorem (Z. 2018+)

If st *is a shuffle-compatible permutation statistic, then there is a homomorphism* Φ_{ist} *on* **Sym** *that can be used to count permutations by the inverse statistic* ist*.*

The Homomorphism Φ_{ides}

- The Hadamard product $*$ on formal power series in t is given by

$$\Big(\sum_{n=0}^{\infty} a_n t^n\Big) * \Big(\sum_{n=0}^{\infty} b_n t^n\Big) = \sum_{n=0}^{\infty} a_n b_n t^n.$$

- Let $\mathbb{Q}[[t*, x]]$ denote the $\mathbb{Q}$-algebra of formal power series in t and x, where the multiplication is Hadamard product in t.

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- Let $\mathbb{Q}[[t*,x]]$ denote the $\mathbb{Q}$-algebra of formal power series in t and x, where the multiplication is Hadamard product in t.

- Define $\Phi_{\mathrm{ides}} \colon \mathbf{Sym} \to \mathbb{Q}[[t*,x]]$ by

$$\Phi_{\mathrm{ides}}(\mathbf{h}_n) = \frac{t}{(1-t)^{n+1}} x^n.$$

Then for $L \vDash n$,

$$\Phi_{\mathrm{ides}}(\mathbf{r}_L) = \sum_{\mathrm{Comp}(\pi)=L} \frac{t^{\mathrm{ides}(\pi)+1}}{(1-t)^{n+1}} x^n.$$

(This works because des is shuffle-compatible.)

Counting $\underline{12\cdots m}$-Avoiders by ides

- Applying Φ_{ides} to $\displaystyle\sum_{\substack{L \\ \text{all parts}<m}} \mathbf{r}_L = \left[\sum_{n=0}^{\infty}(\mathbf{h}_{mn} - \mathbf{h}_{mn+1})\right]^{-1}$ yields:

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Theorem (Z. 2018+)

Let $M^{\text{ides}}_{m,n}(t) := \sum_{\pi\in\text{Av}_n(\underline{12\cdots m})} t^{\text{ides}(\pi)+1}$. *For every* $n \geq 1$ *and* $m \geq 2$, *we have*

$$\sum_{n=0}^{\infty} \frac{M^{\text{ides}}_{m,n}(t)}{(1-t)^{n+1}} x^n = \sum_{n=0}^{\infty} \Bigg[\frac{t}{(1-t)^2} x - \sum_{k=1}^{\infty} \left(\frac{t}{(1-t)^{mk+1}} x^{mk} - \frac{t}{(1-t)^{mk+2}} x^{mk+1} \right) \Bigg]^{*n}.$$

- Problem: This is not a "Hadamard product-free" formula!

Counting $\underline{12\cdots m}$-Avoiders by ides—Take Two!

Theorem (Z. 2018+)

Let $\omega := e^{2\pi i/m}$. For every $n \geq 1$ and $m \geq 2$, we have

$$\sum_{n=0}^{\infty} \frac{M_{m,n}^{\text{ides}}(t)}{(1-t)^{n+1}} x^n = m \sum_{k=0}^{\infty} \left[\sum_{j=1}^{m-1} \frac{1-\omega^{-j}}{(1-\omega^j x)^k} \right]^{-1} t^k$$

$$= \sum_{k=0}^{\infty} \left[\sum_{j=0}^{\infty} \left(\binom{k+jm-1}{k-1} x^{jm} - \binom{k+jm}{k-1} x^{jm+1} \right) \right]^{-1} t^k \quad (\heartsuit)$$

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$$= \sum_{k=0}^{\infty} \left[\sum_{j=0}^{\infty} \left(\binom{k+jm-1}{k-1} x^{jm} - \binom{k+jm}{k-1} x^{jm+1} \right) \right]^{-1} t^k \quad (\heartsuit)$$

- Taking the limit of $(\heartsuit)$ as $m \to \infty$ and extracting coefficients of x^n recovers the classical identity $\dfrac{A_n(t)}{(1-t)^{n+1}} = \sum_{k=0}^{\infty} k^n t^k$.

Further Directions of Research

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Count $\underline{12\cdots m}$-avoiders by inverses of other shuffle-compatible permutation statistics: inverse peak number, inverse left peak number, etc. Do these yield nice Hadamard product-free formulas?

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Do the same for alternating permutations (i.e., $\{\underline{123}, \underline{321}\}$-avoiders).

THANK YOU!