Thresholds of growth rates of sum-closed classes

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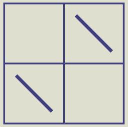
2018 July 12

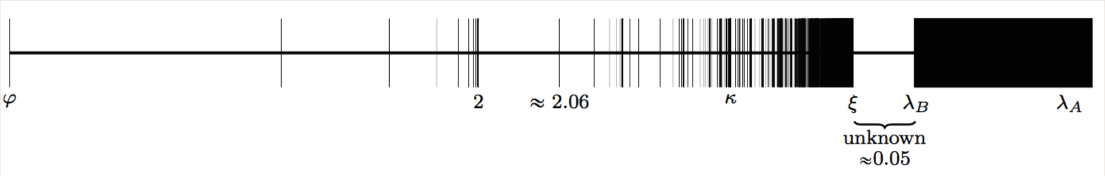
Erratum from PP 2017

- "If $\mathcal{C}$ is a monotone grid class whose corresponding cell graph is a forest, then the growth rate of the centrosymmetric permutations in $\mathcal{C}_{2n}$ equals the growth rate of $\mathcal{C}_n$."
- The proof has a huge flaw, resulting in one direction of inequality instead of equality.
- $k \dots 1$ is centrosymmetric, and it has a gridding on



but it has no centrosymmetric gridding.

Thresholds of growth rates of classes



(Pantone and Vatter, 2016)

- Pantone and Vatter define ξ to be the unique positive root of $x^5 - 2x^4 - x^2 - x - 1$; and $\xi \approx 2.30522$.
- They prove that ξ is the largest number below which there are only countably many growth rates (ξ is the the threshold of uncountability).

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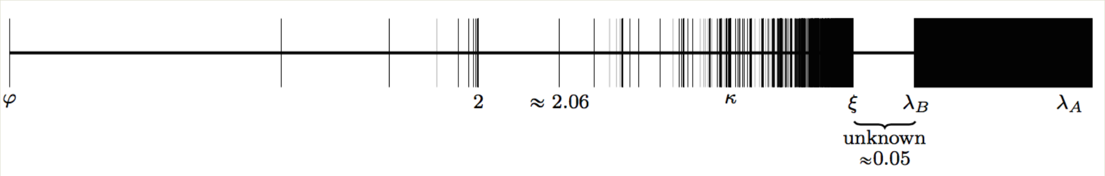

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- They prove that ξ is the largest number below which there are only countably many growth rates (ξ is the the threshold of uncountability).
- A class $\mathcal{C}$ is sum-closed if

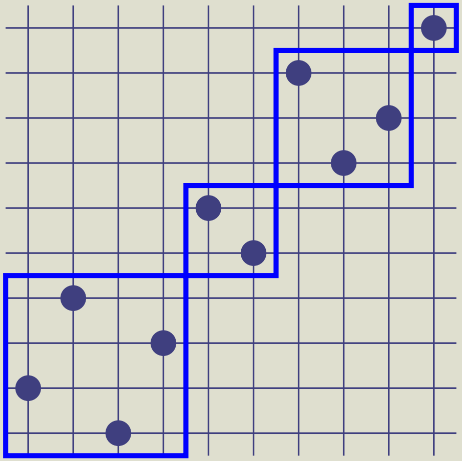
$$\sigma, \tau \in \mathcal{C} \quad \Rightarrow \quad \sigma \oplus \tau \in \mathcal{C}.$$

A. General result on growth rates of sequences
B. Sum-closed classes with growth rate $\leqslant \xi$
C. Threshold of unbounded indecomposables

A. General result on growth rates of sequences

- Let $\mathcal{C}$ be sum-closed, and let $\mathcal{C}^{\not\oplus}$ denote the set of indecomposable permutations in $\mathcal{C}$.
- Let $A(x) = \sum_{n \geqslant 0} |\mathcal{C}_n| x^n$ and $C(x) = \sum_{n \geqslant 1} |\mathcal{C}^{\not\oplus}_n| x^n$.
- Every permutation in $\mathcal{C}$ has a unique decomposition as a sum of indecomposable permutations in $\mathcal{C}$. Therefore,

$$A(x) = \frac{1}{1 - C(x)}.$$



A. General result on growth rates of sequences

(See Flajolet and Sedgewick, Sec. V.2.)

Theorem (T, 2018): For $i \in \{1, 2\}$, let

$$A^{(i)}(x) = \sum_{n \geqslant 1} a_n^{(i)} x^n \quad \text{and} \quad C^{(i)}(x) = \sum_{n \geqslant 1} c_n^{(i)} x^n$$

be formal power series with non-negative real coefficients, with $C^{(i)}(x)$ having zero constant term, satisfying the relation

$$A^{(i)}(x) = \frac{1}{1 - C^{(i)}(x)}.$$

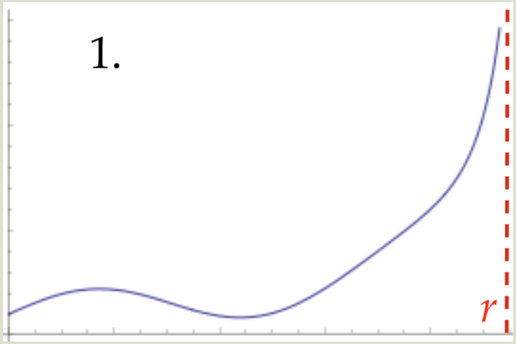
Let r be the radius of convergence of $A^{(1)}$; assume $r > 0$ and $\lim_{x \to r^-} C^{(1)}(x) \geqslant 1$. If $c_n^{(1)} \leqslant c_n^{(2)}$ for all n and $c_n^{(1)} < c_n^{(2)}$ for some n, then $\mathrm{gr}(a_n^{(1)}) < \mathrm{gr}(a_n^{(2)})$.

- Increasing one coefficient of $C(x)$ increases the growth rate of the coefficients of $A(x)$.

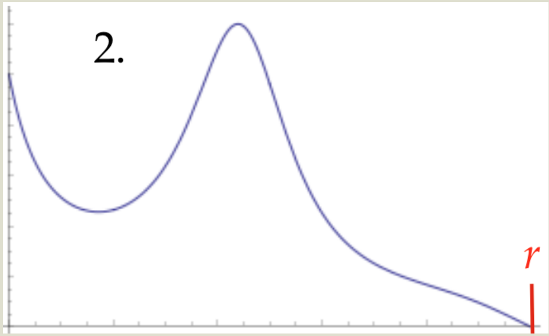
A. General result on growth rates of sequences

(Only one of two cases of the proof is shown here.)

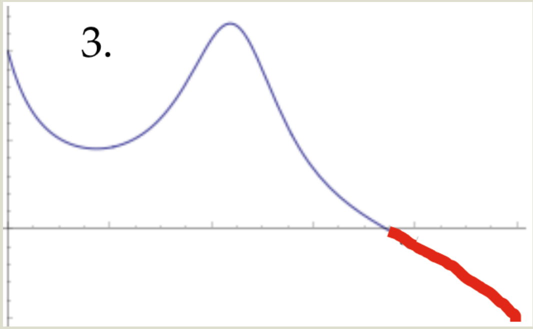
Proof:



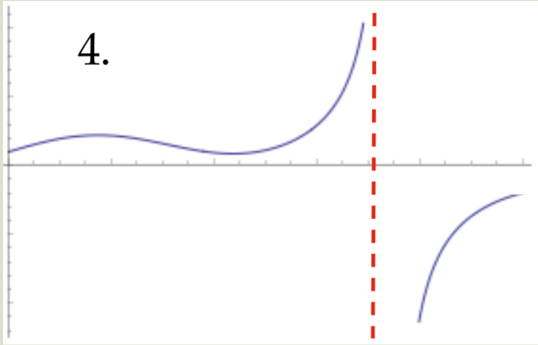
$A^{(1)}(x)$ on $[0, r]$



$1 - C^{(1)}(x) = \dfrac{1}{A^{(1)}(x)}$ on $[0, r]$



$1 - C^{(2)}(x)$ on $[0, r]$



$A^{(2)}(x) = \dfrac{1}{1 - C^{(2)}(x)}$ on $[0, r]$

B. Sum-closed classes with growth rate $\leqslant \xi$

- Let $\xi \approx 2.30522$ be the unique positive root of $x^5 - 2x^4 - x^2 - x - 1$.
- Pantone and Vatter (2016) completely characterize the sum-closed classes with growth rate $\leqslant \xi$ by the enumeration sequence of their indecomposables, *i.e.* $\left(|\mathcal{C}_0^{\not\oplus}|, |\mathcal{C}_1^{\not\oplus}|, |\mathcal{C}_2^{\not\oplus}|, \dots\right)$.

sequence	restriction	growth rate is the greatest real root of
$1,1,2,3,4^i,5,4,1^j$	$i \leqslant 1$, $j \leqslant 5$	$x^{i+j+2}\left(x^5-2x^4-x^2-x-1\right) - x^j\left(x^2-x-3\right)+1$
$1,1,2,3,4^i,5,4,1^j$	i even, $j \leqslant 1$	$x^{i+j+2}\left(x^5-2x^4-x^2-x-1\right) - x^j\left(x^2-x-3\right)+1$
$1,1,2,3,4^i,5,3,3,2$	$i=1$ or i even	$x^{i+4}\left(x^5-2x^4-x^2-x-1\right) - x^4+2x^3+x+2$
$1,1,2,3,4^i,5,3,3,1^\infty$	$i \leqslant 1$	$x^{i+3}\left(x^5-2x^4-x^2-x-1\right) - x^3+2x^2+2$
$1,1,2,3,4^i,5,3,3,1^j$	$i \leqslant 1$	$x^{i+j+3}\left(x^5-2x^4-x^2-x-1\right) - x^j\left(x^3-2x^2-2\right)+1$
$1,1,2,3,4^i,5,3,3,1^j$	i even, $j \leqslant 1$	$x^{i+j+3}\left(x^5-2x^4-x^2-x-1\right) - x^j\left(x^3-2x^2-2\right)+1$
$1,1,2,3,4^i,5,3^j,2^\infty$	$i=1$ or i even, $j \leqslant 1$	$x^{i+j+1}\left(x^5-2x^4-x^2-x-1\right) - x^j(x-2)+1$
$1,1,2,3,4^i,5,3^j,2^k,1^\infty$	$i \leqslant 1$, $j \leqslant 1$	$x^{i+j+k+1}\left(x^5-2x^4-x^2-x-1\right) - x^{j+k}(x-2)+x^k+1$
$1,1,2,3,4^i,5,3^j,2^k,1^\ell$	$i \leqslant 1$, $j \leqslant 1$	$x^{i+j+k+\ell+1}\left(x^5-2x^4-x^2-x-1\right) - x^{j+k+\ell}(x-2)+x^{k+\ell}+x^\ell+1$
$1,1,2,3,4^i,5,3^j,2^k,1^\ell$	i even, $j \leqslant 1$, $\ell \leqslant 1$	$x^{i+j+k+\ell+1}\left(x^5-2x^4-x^2-x-1\right) - x^{j+k+\ell}(x-2)+x^{k+\ell}+x^\ell+1$
$1,1,2,3,4^i,3^\infty$		$x^i\left(x^5-2x^4-x^2-x-1\right)+1$
$1,1,2,3,4^i,3^j,2^\infty$		$x^{i+j}\left(x^5-2x^4-x^2-x-1\right)+x^j+1$
$1,1,2,3,4^i,3^j,2^k,1^\infty$		$x^{i+j+k}\left(x^5-2x^4-x^2-x-1\right)+x^{j+k}+x^k+1$
$1,1,2,3,4^i,3^j,2^k,1^\ell$		$x^{i+j+k+\ell}\left(x^5-2x^4-x^2-x-1\right)+x^{j+k+\ell}+x^{k+\ell}+x^\ell+1$

- We explore some other consequences of this work.

B. Sum-closed classes with growth rate $\leqslant \xi$

Results essentially in Pantone and Vatter, 2016:

- **Prop.** Every sum-closed $\mathcal{C}$ with $\mathrm{gr}(\mathcal{C}) \leqslant \xi$ has $|\mathcal{C}_n^{\not\oplus}| \leqslant 5$ for all n, and ξ is the largest such number.
  - In my proof, the growth-rate theorem is used in the case of $|\mathrm{gr}(\mathcal{C})| = \xi$.
- **Prop.** Every sum-closed $\mathcal{C}$ with $\mathrm{gr}(\mathcal{C}) \leqslant \xi$ has a rational generating function, and ξ is the largest such number.
  - In contrast, there are uncountably many classes with growth rate $\kappa \approx 2.20577$ whose generating functions are not even D-algebraic (Albert, Ruškuc, and Vatter, 2015).

B. Sum-closed classes with growth rate $\leqslant \xi$

Prop. Every sum-closed $\mathcal{C}$ with $\mathrm{gr}(\mathcal{C}) \leqslant \xi$ has $|\mathcal{C}_n^{\not\oplus}| \leqslant 5$ for all n, and ξ is the largest such number.

Proof idea:

- Pantone and Vatter have "tapering" results on the sequence $|\mathcal{C}_n^{\not\oplus}|$ (for any class $\mathcal{C}$).
- We use these in conjunction with their examples of potential sequences that lead to growth rate $\geqslant \xi$:

sequence	growth rate is the greatest real root of
$1,1,2,3,4^\infty$	$x^5 - 2x^4 - x^2 - x - 1$
$1,1,2,3,4^i,5,3,3,2,1$	$x^5 - 2x^4 - x^2 - x - 1$
$1,1,2,3,4^i,5,3,3,3$	$x^{i+4}(x^5 - 2x^4 - x^2 - x - 1) - x^4 + 2x^3 + 3$
$1,1,2,3,4^i,5,4,1,1,1,1,1,1$	$x^{i+8}(x^5 - 2x^4 - x^2 - x - 1) - x^8 + x^7 + 3x^6 + 1$
$1,1,2,3,4^i,5,4,2$	$x^{i+3}(x^5 - 2x^4 - x^2 - x - 1) - x^3 + x^2 + 2x + 2$
$1,1,2,3,4^i,5,5$	$x^{i+2}(x^5 - 2x^4 - x^2 - x - 1) - x^2 + 5$
$1,1,2,3,4^i,6,2,1,1$	$x^{i+4}(x^5 - 2x^4 - x^2 - x - 1) - 2x^4 + 4x^3 + x^2 + 1$
$1,1,2,3,4^i,6,2,2$	$x^{i+3}(x^5 - 2x^4 - x^2 - x - 1) - 2x^3 + 4x^2 + 2$
$1,1,2,3,4^i,6,3$	$x^{i+2}(x^5 - 2x^4 - x^2 - x - 1) - 2x^2 + 3x + 3$
$1,1,2,3,4^i,7,1$	$x^{i+2}(x^5 - 2x^4 - x^2 - x - 1) - 3x^2 + 6x + 1$
$1,1,2,3,4^i,8$	$x^{i+1}(x^5 - 2x^4 - x^2 - x - 1) - 4x + 8$

- If the sequence $|\mathcal{C}_n^{\not\oplus}|$ dominates one of those, then the growth rate is $> \xi$.

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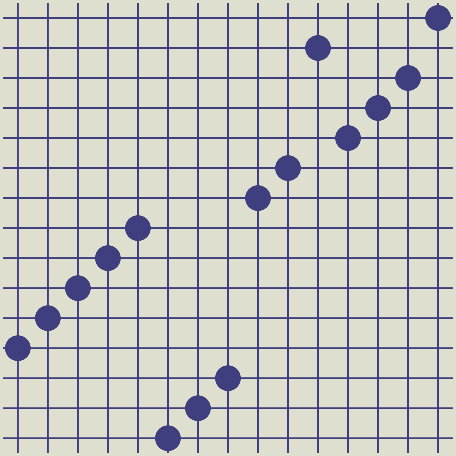
- Consider $\mathcal{C} = \mathrm{Av}(321, 3142, 2413)$.



- $|\mathcal{C}_n^{\not\oplus}| = n - 1$ and $\mathrm{gr}(\mathcal{C}) = \tau \approx 2.32472$, the unique positive root of $x^3 - 3x^2 + 2x - 1$.

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  - This is because $C(x) = x + \dfrac{x^2}{(1-x)^2}$, so $A(x) = \dfrac{1}{1 - C(x)} = \dfrac{(1-x)^2}{1 - 3x + 2x^2 - x^3}$, and $1/\tau$ is the smallest singularity of $A(x)$.

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- **Conjecture.** If $\mathcal{C}$ is sum-closed and $|\mathcal{C}_n^{\not\oplus}|$ is unbounded, then $|\mathcal{C}_n^{\not\oplus}| \geqslant n - 1$.

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- **Conjecture.** If $\mathcal{C}$ is sum-closed and $|\mathcal{C}_n^{\not\oplus}|$ is unbounded, then $|\mathcal{C}_n^{\not\oplus}| \geqslant n - 1$.
- **Conjollary.** Every sum-closed $\mathcal{C}$ with $\mathrm{gr}(\mathcal{C}) < \tau$ has bounded indecomposables, and τ is the largest such number.

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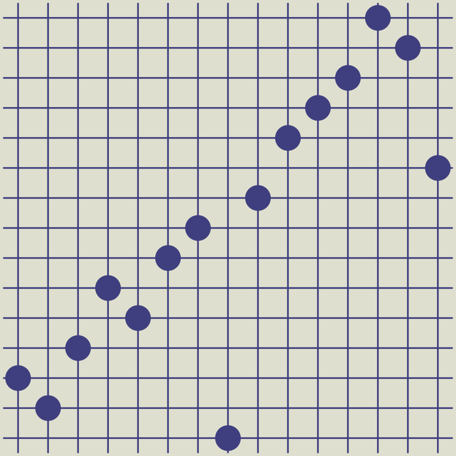

- $|\mathcal{C}_n^{\not\oplus}| = f_n$ and $\mathrm{gr}(\mathcal{C}) = 1 + \sqrt{2} \approx 2.41421$.

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- This puts the threshold of exponential indecomposables between ξ and $1 + \sqrt{2}$.

Further questions

- For each $m \geqslant 6$, what is the largest growth rate below which the number of indecomposables is $\leqslant m$?
- Is there sum-closed $\mathcal{C}$ with $\overline{\mathrm{gr}}(\mathcal{C}^{\not\oplus})$ strictly between 1 and ϕ?
- Does $\mathcal{C}^{\not\oplus}$ always have a proper growth rate?
  - The question for $\mathcal{C}^{\not\oplus}$ may be easier than for $\mathcal{C}$, because it suffices to consider sum-closed $\mathcal{C}$.