Stack Sorting r-tiers

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We consider a second algorithm *stack sorting with reverse passes* on single stack where outputting the correct element takes priority over the order of elements within the stack.

Specifically, we apply a sorting algorithm on a stack whereby the entries of the permutation are pushed into the stack in the usual way. We pop entries from the stack only if they are the next needed entry for the output.

If entries remain in the stack, we restart the process with these remaining entries in the reverse of their original order.

Definition

We will call each repetition of the stack sorting algorithm a **reverse pass**.

First reverse pass

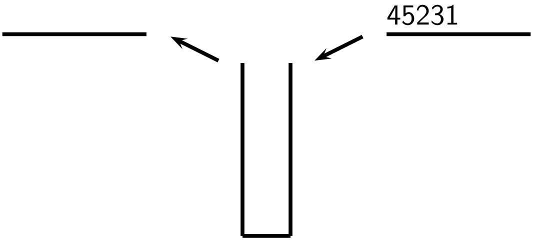
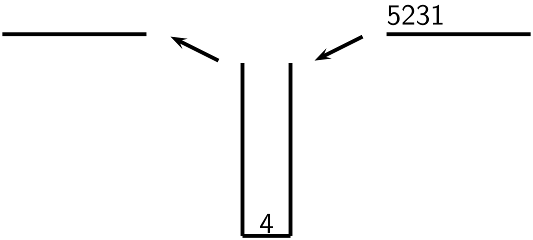


Figure: Sorting 45231 with two reverse passes

First reverse pass



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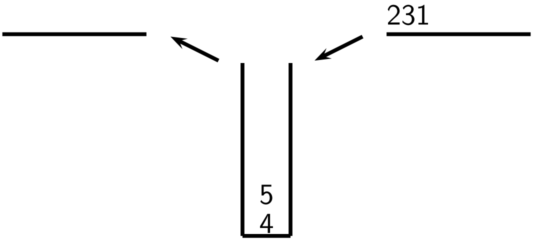

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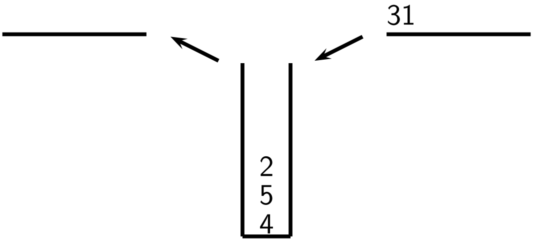

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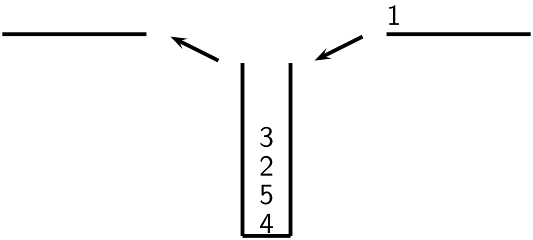

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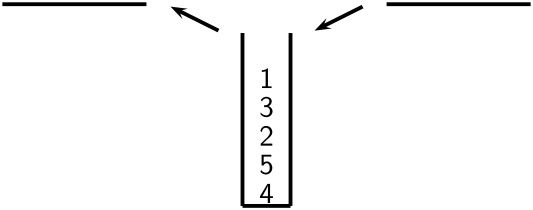

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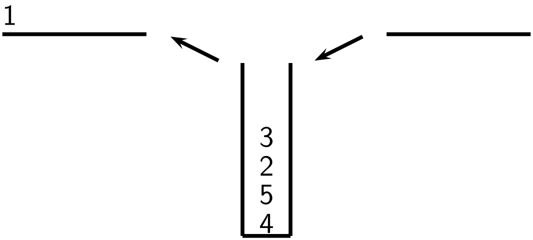

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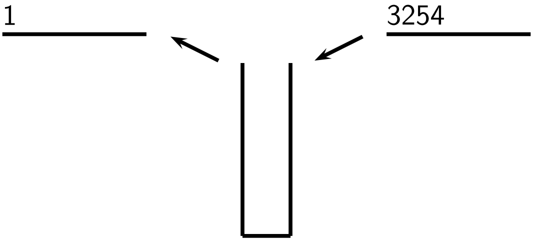

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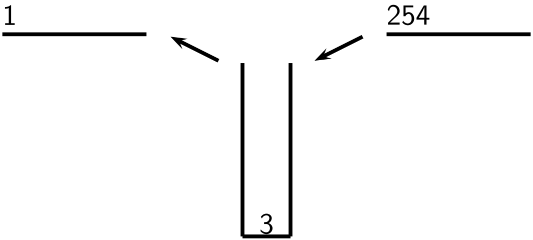

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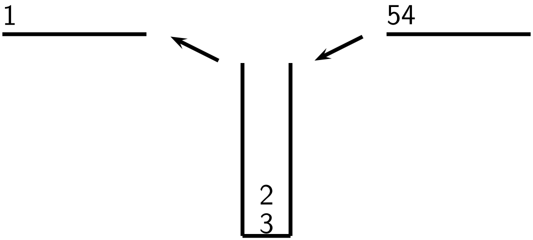

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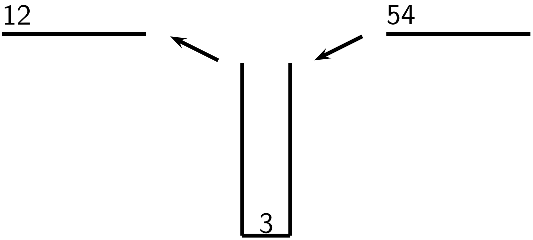

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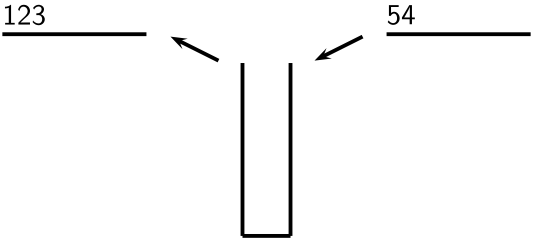

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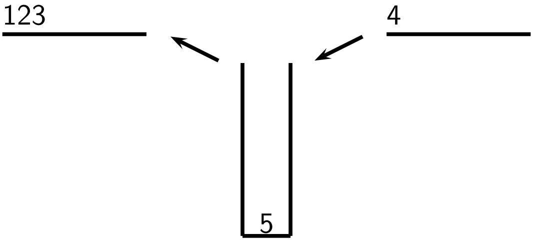

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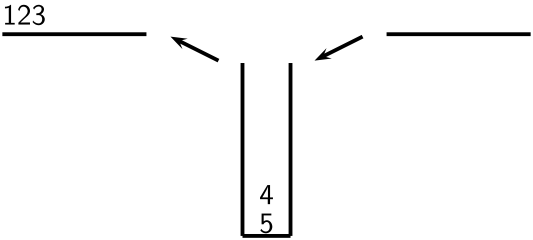

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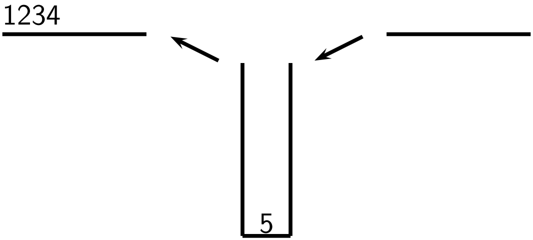

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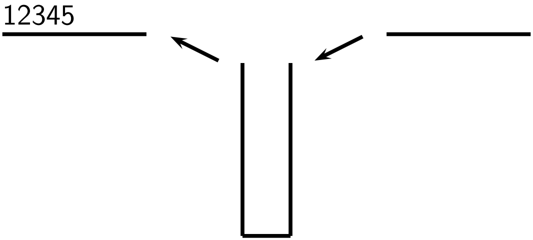

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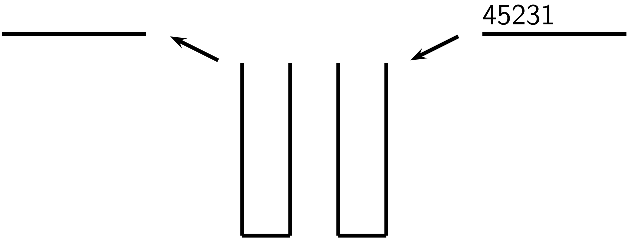
However, the reversal makes this algorithm acts more like a machine of stacks in series.



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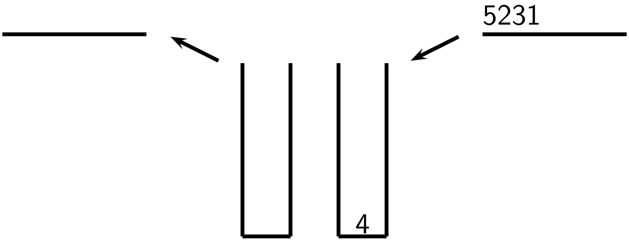

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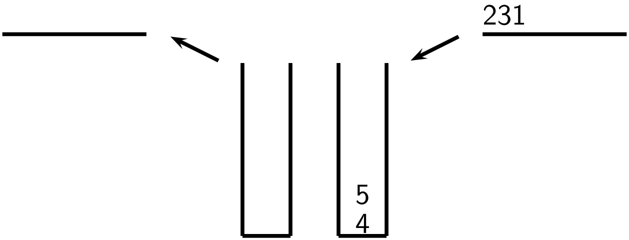

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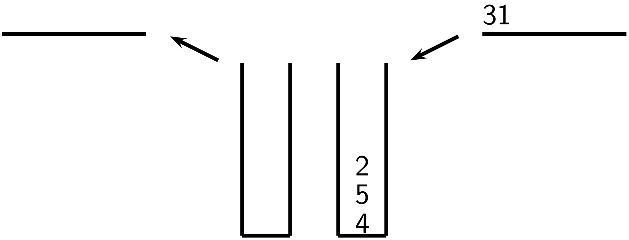

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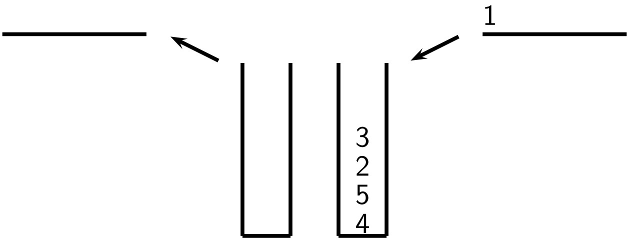

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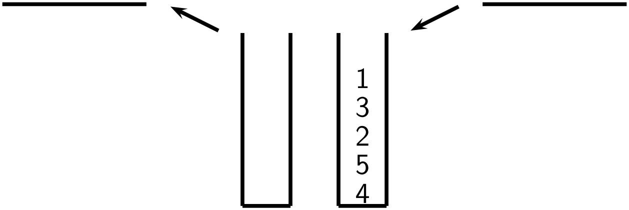

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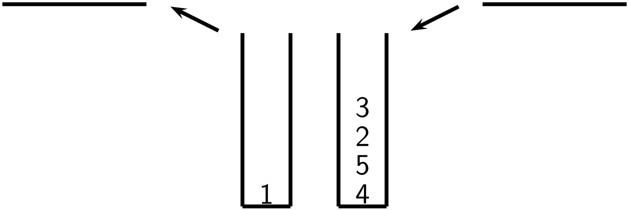

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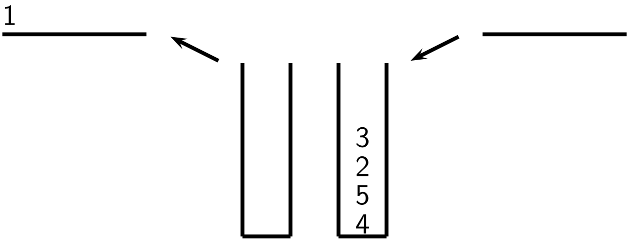

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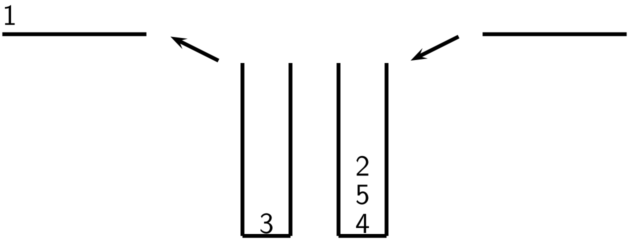

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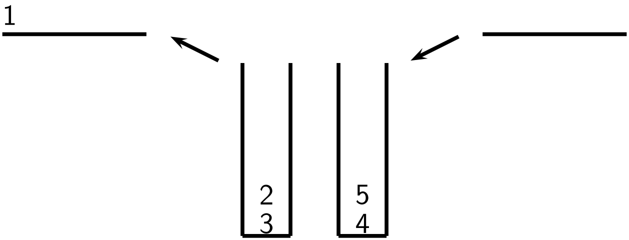

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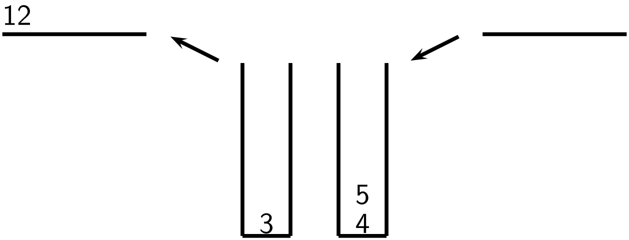

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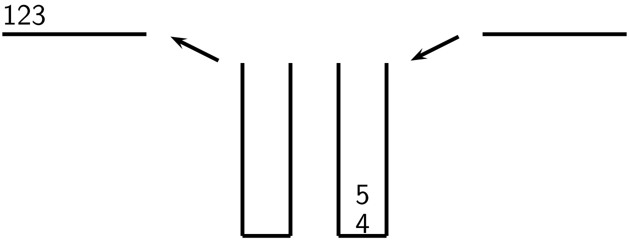

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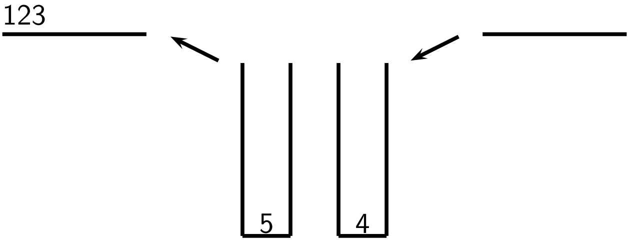

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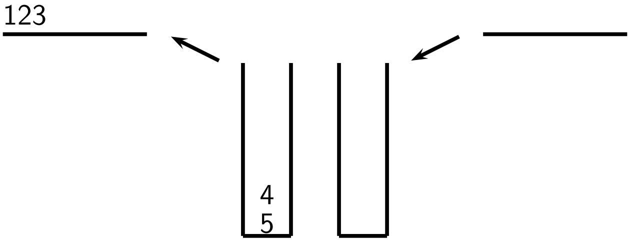

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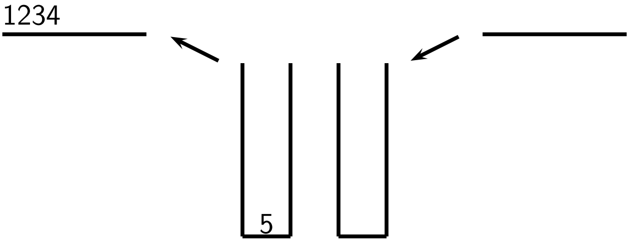

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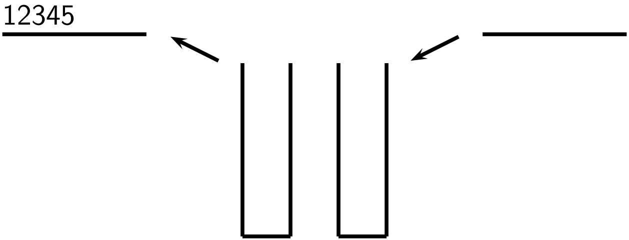

Figure: Sorting 45231.

Definition

We say the r-**tier** of a permutation σ is the number of times the algorithm must restart in order to sort σ.

Further any permutation with r-tier of at most r is said to be $(r+1)$-reverse-pass sortable.

Example

The permutation 231 has r-tier 1.
All other permutations in S_3 have r-tier 0.
So we can say all permutations in S_3 are 2-reverse-pass sortable.

As before the notion of separated pairs (i.e. covincular patterns) will play an important role.

Definition

Let $\sigma \in S_n$ and let $i \in \{1, 2, ...n-1\}$. Call $(i, i+1)$ **separated** if there is a $k > i+1$ which occurs between i and $i+1$ in σ.

We say the separated pair $(i, i+1)$ is in an **up** orientation if i precedes $i+1$ in σ and $(i, i+1)$ is in a **down** orientation if i follows $i+1$ in σ.

Note that an up separated pair is the covincular pattern 132 where the $1, 2$ are adjacent in value. Similarly, a down separated pair is the covincular pattern 231 where again the $1, 2$ are adjacent in value.

Theorem

The r-tier of a permutation under this sorting algorithm is exactly the maximum number r of separated pairs $(i_i, i_1+1), (i_2, i_2+1), \ldots, (i_r, i_r+1)$ *where* $i_1 < i_2 < i_3 < ... < i_r$ *and such that the orientations of the pairs alternate (in this order) beginning with a down separated pair.*

Corollary

The class of $(r+1)$*-reverse pass sortable permutations form a permutation class and this class has a finite basis.*

Recall the class of permutations that are 2-pass sortable have basis

$$B = \{24153, 24513, 24531, 34251, 35241, 42513,$$

$$42531, 45231, 261453, 231564, 523164\}.$$

Theorem

A permutation has r-tier r ≤ 1 *(i.e. is* 2*-reverse-pass sortable) if and only if it avoids the permutations* $2413, 2431,$ *and* 23154*.*

Theorem

A permutation has r-tier $r \leq 1$ (i.e. is 2-reverse-pass sortable) if and only if it avoids the permutations $2413, 2431,$ *and* 23154.

Theorem

A permutation has r-tier $r \leq 2$ (i.e. is 3-reverse-pass sortable) if and only if it avoids the permutations

$$24153, 24513, 24531, 42513, 42531,$$

$$231564, 261453, 523164, 562413, 562431,$$

$$6723154.$$

	r = 0	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6	r = 7	r = 8
n = 1	1								
n = 2	2								
n = 3	5	1							
n = 4	14	8	2						
n = 5	42	47	26	5					
n = 6	132	248	228	96	16				
n = 7	429	1249	1702	1178	421	61			
n = 8	1430	6154	11704	11840	6816	2102	272		
n = 9	4862	30013	76845	106567	88020	43347	11841	1385	
n = 10	16796	145764	490866	896560	997056	697644	302002	74176	7936

Table: Number of permutations of length *n* and exact *r*-tier *r*

Notation

Let $\rho(n)$ represent the maximum r-tier of any permutation of length n.

Proposition

For any integer $n \geq 3$ we have $\rho(n) = \rho(n-1) + 1 = n - 2$.

As an immediate consequence, any permutation of length n and r-tier $\rho(n)$ has $(1,2), (2,3), \ldots, (n-2, n-1)$ as its maximum length alternating sequence of separated pairs where $(1,2)$ has down orientation.

	r = 0	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6	r = 7	r = 8
n = 1	1								
n = 2	2								
n = 3	5	1							
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Table: Number of permutations of length n and exact r-tier r

Definition

The **Euler down-up permutations** are alternating permutations that begin with a descent and are enumerated by (one of the sequences called) the Euler numbers E_n.

Example

The Euler down-up permutations of length 4 are $2143, 3142, 3241, 4132, 4231$, so $E_4 = 5$.

Looking this sequence (A001250) we have some familiar terms:

E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9
1	2	5	16	61	272	1385	7936

Definition

The **Entringer number** $E_{n,k}$ is the number of down/up permutations of $[n]$ beginning with k. Thus $\sum_{k=1}^{n} E_{n,k} = E_n$.

Let $\mathcal{E}_{n,k}$ be the set of alternating permutations of length n beginning with k.

Theorem

(Entringer, Seidel)
The sequence $\{E_{n,k}\}$ *where* $1 \le k \le n$ *is defined recursively where* $E_{1,1} = 1$, $E_{n,1} = 0$ *when* $n > 1$ *and*

$$E_{n,k} = E_{n,k-1} + E_{n-1,n+1-k}.$$

Example

Consider the down/up permutations of length five. Note $E_{5,5} = 5$ as it counts

$$51324, 51423, 52314, 52413, 53412.$$

By exchanging the 4 and 5 in the previous five permutations, we have $E_{5,4} = 5$ since it counts

$$41325, 41523, 42315, 42513, 43512.$$

Then $E_{5,3} = 4$, counting

$$31425, 31524, 32415, 32514,$$

and finally $E_{5,2} = 2$ as it counts

$$21435, 21534.$$

Note $E_{5,1} = 0$. Collecting all of these permutations we see $E_5 = 16$.

Definition

Let R_n to be number of permutations of length n with r-tier $r = n - 2$.

Further, define $R_{n,k}$ to be the number of permutations of length n with r-tier $r = n - 2$ where 1 is in position $k + 1$.

Finally, let $\mathcal{R}_{n,k}$ be the set of permutations of length n with r-tier $r = n - 2$.

Example

Consider the permutations of length six with r-tier four. Note $R_{6,5} = 5$ as it counts

$$246351, 246531, 426351, 426531, 462531.$$

By moving the 1 one position to the left, we have $R_{6,4} = 5$ since it counts

$$246315, 246513, 426315, 426513, 462513.$$

Then $R_{6,3} = 4$, counting

$$246135, 246153, 426135, 426153,$$

and finally $R_{6,2} = 2$ as it counts

$$241635, 241653.$$

Note $R_{6,6} = R_{6,1} = 0$. Thus $R_6 = 16$.

Theorem

The permutations of length $n \geq 3$ with r-tier $n-2$ are an Entringer family. Specifically, $R_{n,k} = E_{n-1,k}$ for $1 \leq k \leq n-1$ and so $R_n = E_{n-1}$ when $n \geq 3$.

Proof.

$Sm(i) =$ the number of inversions of π where π_i is the first element.

We construct a bijection $f \colon \mathcal{E}_{n-1,k} \to \mathcal{R}_{n,k}$ defining $f(\pi)$ by beginning at π_1 and proceeding left to right as follows:

1. Place each odd element $2i+1$ in position $Sm(2i+1)+2$ *of the remaining positions*.
2. Place each even element $2i$ in position $Sm(2i)+1$ *of the remaining positions*.
3. Element n gets the remaining spot.

□

Example

Consider an alternating permutation $\pi = 6572314$ which is one of the permutations counted by $\mathcal{E}_{7,6}$. Then $\sigma = f(\pi)$ is such that:

1 is in the $Sm(1) + 2 = 7$th position of the open positions in σ

That is, the steps described give us:
$\sigma = \_\_ \_\_ \_\_ \_\_ \_\_ \_\_ \; 1 \; \_\_$

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1 is in the $Sm(1) + 2 = 7$th position of the open positions in σ

2 is in the $Sm(2) + 1 = 5$th position of the open positions in σ

That is, the steps described give us:
$\sigma = \_\_ \; \_\_ \; \_\_ \; \_\_ \; 2 \; \_\_ \; 1 \; \_\_$

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2 is in the $Sm(2) + 1 = 5$th position of the open positions in σ

3 is in the $Sm(3) + 2 = 6$th position of the open positions in σ

That is, the steps described give us:

$\sigma = \_\_ \; \_\_ \; \_\_ \; \_\_ \; 2 \; \_\_ \; 1 \; 3$

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4 is in the $Sm(4) + 1 = 2$nd position of the open positions in σ

That is, the steps described give us:

$\sigma = \_\_\ 4\ \_\_\ \_\_\ 2\ \_\_\ 1\ 3$

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4 is in the $Sm(4) + 1 = 2$nd position of the open positions in σ

5 is in the $Sm(5) + 2 = 3$rd position of the open positions in σ

That is, the steps described give us:

$\sigma = \_\_\ 4\ \_\_\ 5\ 2\ \_\_\ 1\ 3$

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5 is in the $Sm(5) + 2 = 3$rd position of the open positions in σ

6 is in the $Sm(6) + 1 = 1$st position of the open positions in σ

That is, the steps described give us:
$\sigma = 6\,4\,\_\_\,5\,2\,\_\_\,1\,3$

Example

Consider an alternating permutation $\pi = 6572314$ which is one of the permutations counted by $\mathcal{E}_{7,6}$. Then $\sigma = f(\pi)$ is such that:

1 is in the $Sm(1) + 2 = 7$th position of the open positions in σ

2 is in the $Sm(2) + 1 = 5$th position of the open positions in σ

3 is in the $Sm(3) + 2 = 6$th position of the open positions in σ

4 is in the $Sm(4) + 1 = 2$nd position of the open positions in σ

5 is in the $Sm(5) + 2 = 3$rd position of the open positions in σ

6 is in the $Sm(6) + 1 = 1$st position of the open positions in σ

7 is in the $Sm(7) + 2 = 2$nd position of the open positions in σ

That is, the steps described give us:
$\sigma = 6\ 4\ \_\_\ 5\ 2\ 7\ 1\ 3$

Example

Consider an alternating permutation $\pi = 6572314$ which is one of the permutations counted by $\mathcal{E}_{7,6}$. Then $\sigma = f(\pi)$ is such that:

1 is in the $Sm(1) + 2 = 7$th position of the open positions in σ

2 is in the $Sm(2) + 1 = 5$th position of the open positions in σ

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4 is in the $Sm(4) + 1 = 2$nd position of the open positions in σ

5 is in the $Sm(5) + 2 = 3$rd position of the open positions in σ

6 is in the $Sm(6) + 1 = 1$st position of the open positions in σ

7 is in the $Sm(7) + 2 = 2$nd position of the open positions in σ

The position of 8 in σ is the only open position in σ

That is, the steps described give us:
$\sigma = 64852713$

$L(i) =$ the number of inversions of σ where i is the second element.

Example

Consider a permutation $\sigma = 6247153$ with r-tier which is one of the permutations counted by $\mathcal{R}_{7,4}$. Then $\pi = f^{-1}(\sigma)$ is such that:

π_1 is the $L(1) = 4$th largest of the remaining values of $\pi,$ i.e. $\pi_1 = 4.$

That is, the steps described give us
$f^{-1}(\sigma) = 4$

$L(i) =$ the number of inversions of σ where i is the second element.

Example

Consider a permutation $\sigma = 6247153$ with r-tier which is one of the permutations counted by $\mathcal{R}_{7,4}$. Then $\pi = f^{-1}(\sigma)$ is such that:

π_1 is the $L(1) = 4$th largest of the remaining values of π, i.e. $\pi_1 = 4$.

π_2 is the $L(2) + 1 = 2$nd largest of the remaining values of π, i.e. $\pi_2 = 2$.

That is, the steps described give us
$f^{-1}(\sigma) = 42$

$L(i) =$ the number of inversions of σ where i is the second element.

Example

Consider a permutation $\sigma = 6247153$ with r-tier which is one of the permutations counted by $\mathcal{R}_{7,4}$. Then $\pi = f^{-1}(\sigma)$ is such that:

π_1 is the $L(1) = 4$th largest of the remaining values of $\pi,$ i.e. $\pi_1 = 4.$

π_2 is the $L(2) + 1 = 2$nd largest of the remaining values of $\pi,$ i.e. $\pi_2 = 2.$

π_3 is the $L(3) = 4$th largest of the remaining values of $\pi,$ that is $\pi_3 = 6.$

That is, the steps described give us
$f^{-1}(\sigma) = 426$

$L(i) =$ the number of inversions of σ where i is the second element.

Example

Consider a permutation $\sigma = 6247153$ with r-tier which is one of the permutations counted by $\mathcal{R}_{7,4}$. Then $\pi = f^{-1}(\sigma)$ is such that:

π_1 is the $L(1) = 4$th largest of the remaining values of $\pi,$ i.e. $\pi_1 = 4.$

π_2 is the $L(2) + 1 = 2$nd largest of the remaining values of $\pi,$ i.e. $\pi_2 = 2.$

π_3 is the $L(3) = 4$th largest of the remaining values of $\pi,$ that is $\pi_3 = 6.$

π_4 is the $L(4) + 1 = 2$nd largest of the remaining values of $\pi,$ that is $\pi_4 = 3.$

That is, the steps described give us
$f^{-1}(\sigma) = 4263$

$L(i) =$ the number of inversions of σ where i is the second element.

Example

Consider a permutation $\sigma = 6247153$ with r-tier which is one of the permutations counted by $\mathcal{R}_{7,4}$. Then $\pi = f^{-1}(\sigma)$ is such that:

π_1 is the $L(1) = 4$th largest of the remaining values of π, i.e. $\pi_1 = 4$.

π_2 is the $L(2) + 1 = 2$nd largest of the remaining values of π, i.e. $\pi_2 = 2$.

π_3 is the $L(3) = 4$th largest of the remaining values of π, that is $\pi_3 = 6$.

π_4 is the $L(4) + 1 = 2$nd largest of the remaining values of π, that is $\pi_4 = 3$.

π_5 is the $L(5) = 2$nd largest of the remaining values of π, that is $\pi_5 = 5$.

That is, the steps described give us
$f^{-1}(\sigma) = 42635$

$L(i) =$ the number of inversions of σ where i is the second element.

Example

Consider a permutation $\sigma = 6247153$ with r-tier which is one of the permutations counted by $\mathcal{R}_{7,4}$. Then $\pi = f^{-1}(\sigma)$ is such that:

π_1 is the $L(1) = 4$th largest of the remaining values of π, i.e. $\pi_1 = 4$.

π_2 is the $L(2) + 1 = 2$nd largest of the remaining values of π, i.e. $\pi_2 = 2$.

π_3 is the $L(3) = 4$th largest of the remaining values of π, that is $\pi_3 = 6$.

π_4 is the $L(4) + 1 = 2$nd largest of the remaining values of π, that is $\pi_4 = 3$.

π_5 is the $L(5) = 2$nd largest of the remaining values of π, that is $\pi_5 = 5$.

π_6 is the $L(6) + 1 = 1$st largest of the remaining values of π, that is $\pi_6 = 1$.

That is, the steps described give us
$f^{-1}(\sigma) = 426351$.

	r = 0	r ≤ 1	r ≤ 2	r ≤ 3	r ≤ 4	r ≤ 5	r ≤ 6	r ≤ 7
n = 1	1	1	1	1	1	1	1	1
n = 2	2	2	2	2	2	2	2	2
n = 3	5	6	6	6	6	6	6	6
n = 4	14	22	24	24	24	24	24	24
n = 5	42	89	115	120	120	120	120	120
n = 6	132	380	608	704	720	720	720	720
n = 7	429	1678	3380	4558	4979	5040	5040	5040
n = 8	1430	7584	19288	31128	37946	40048	40320	40320
n = 9	4862	34875	111720	218287	306307	349654	361495	362880
n = 10	16796	162560	653426	1549986	2547042	3244686	3546688	3620864

Table: Number of permutations of length n and r-tier at most r

Conjecture

$\mathrm{Av}(2413, 2431, 23154)$ *is Wilf-equivalent to* $\mathrm{Av}(4321, 4213)$.

Thank you!