Recognizing merge classes

Generalized coloring of permutations

Michal Opler
Joint work with Vít Jelínek and Pavel Valtr

Computer Science Institute of Charles University in Prague

Permutation Patterns
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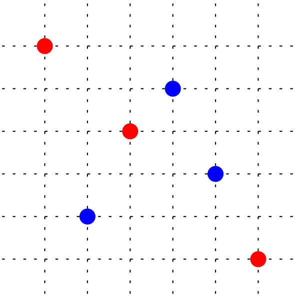
Merges

Definition

Permutation π is a merge of permutations σ and τ if the elements of π can be colored red and blue, so that the red elements are a copy of σ and the blue ones of τ.



One possible merge of 132 and 321 is 624531.

Recognition problems

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For two permutation classes $\mathcal{C}$ and $\mathcal{D}$, let $\mathcal{C} \odot \mathcal{D}$ be the class of permutations obtained by merging a $\sigma \in \mathcal{C}$ with a $\tau \in \mathcal{D}$.

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For a permutation class $\mathcal{C}$, $\mathcal{C}$-recognition is the decision problem to determine whether a given permutation belongs to $\mathcal{C}$.

Example: $\text{Av}(k \dots 1)$-recognition is the usual coloring problem.

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Question

How hard is the $(\mathcal{C} \odot \mathcal{D})$-recognition problem for various choices of $\mathcal{C}$ and $\mathcal{D}$.

Generalized coloring of graphs

Definition

For a fixed k-tuple $\mathcal{G}_1, \ldots, \mathcal{G}_k$ of graph classes, a generalized coloring of a graph is an assignment of colors $1, 2, \ldots, k$ to its vertices so that the vertices of color i induce a subgraph from $\mathcal{G}_i$.

In particular, if all the $\mathcal{G}_i$ are equal to the class of edgeless graphs, this notion reduces to the classical notion of k-coloring.

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In particular, if all the $\mathcal{G}_i$ are equal to the class of edgeless graphs, this notion reduces to the classical notion of k-coloring.

Theorem (Farrugia)

If all the $\mathcal{G}_i$ are hereditary and additive (i.e., closed under taking induced subgraphs and forming disjoint unions) then the problem is NP-hard, except the trivially polynomial case when $k = 2$ and both $\mathcal{G}_1$ and $\mathcal{G}_2$ are equal to the class of edgeless graphs.

Prior results

Facts

($\mathcal{C} \odot \mathcal{D}$)-recognition is tractable, i.e. polynomially solvable, whenever

- *$\mathcal{C}$-recognition is tractable and $\mathcal{D}$ is finite,*
- $\mathcal{C} = Av(1 \ldots k)$ *and* $\mathcal{D} = Av(1 \ldots l)$, *and*
- $\mathcal{C} = Av(1 \ldots k)$ *and* $\mathcal{D} = Av(l \ldots 1)$.

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($\mathcal{C} \odot \mathcal{D}$)-recognition is tractable, whenever $\mathcal{C} \odot \mathcal{D}$ has a finite basis.

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Theorem (Ekim et al.)

There is a polynomial algorithm for

- *$(\mathcal{L} \odot Av(21))$-recognition and*
- *$(\mathcal{L} \odot \overline{\mathcal{L}})$-recognition, where*

$\mathcal{L}$ is the class of layered permutations and $\overline{\mathcal{L}}$ is the class of co-layered permutations.

Online recognition motivation

Problem: Decide whether π of length n belongs to $\mathrm{Av}(12) \odot \mathrm{Av}(21)$.

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Moreover $S_0 = \{(+\infty, -\infty)\}$.

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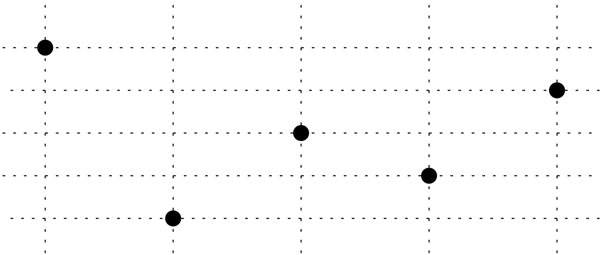
Observation

π belongs to $\text{Av}(12) \odot \text{Av}(21)$ if and only if S_n is not empty.

Algorithm: Incrementally compute $S_1, \ldots, S_n$.



π_i		5	1	3	2	4
S_i	$(+\infty, -\infty)$	$(5, -\infty)$ $(+\infty, 5)$	$(1, -\infty)$ $(5, 1)$ $(1, 5)$	$(1, 3)$ $(5, 3)$ $(3, 1)$	$(2, 3)$ $(3, 2)$ $(2, 1)$	$(2, 4)$ $(3, 4)$

Run of the algorithm on permutation 51324.

Nondeterministic point of view

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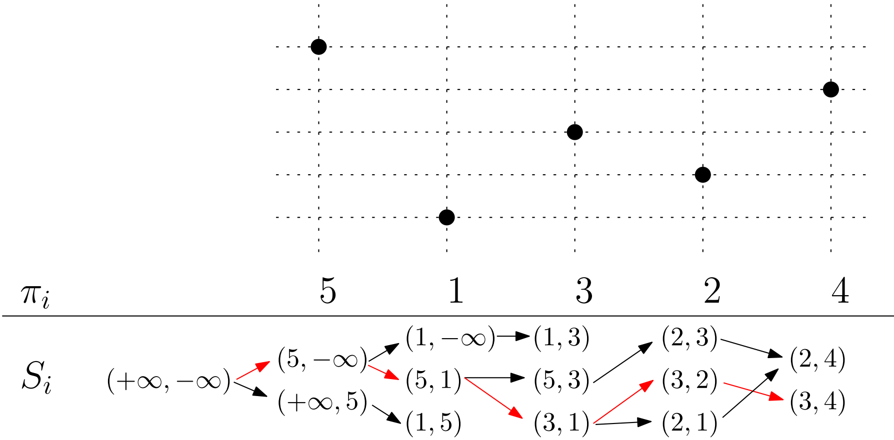
Suppose we are trying to solve the same problem nondeterministically. Let A be a nondeterministic algorithm that stores in memory only pair of the last values in each sequence.

- A initializes its memory state to $(+\infty, -\infty)$.
- For $k = 1 \dots n$
  - A guesses whether π_k belongs to increasing or decreasing sequence, and
  - updates the pair of last values or terminates unsuccessfully.
  - A accepts π if there is valid pair in its memory after receiving π_n.

Observation

There is an accepting computation of A on π if and only if π belongs to $\text{Av}(12) \odot \text{Av}(21)$.



Possible run of the nondeterministic algorithm that decomposes 51324 into 53 and 124.

NLOL-recognition

We say that a nondeterministic algorithm A is a nondeterministic logspace on-line recognizer of $\mathcal{C}$ if it recognizes $\mathcal{C}$ in the following setting:

- A receives an integer n, then
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- Afterwards, A answers whether $\pi_1, \ldots, \pi_k$ is order-isomorphic to some $\pi' \in \mathcal{C}$.
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Observation

For any permutation class NLOL-recognizable class $\mathcal{C}$, the $\mathcal{C}$-recognition problem is tractable.

Properties of NLOL-recognizable classes

Lemma

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(d) *The classes $\mathcal{C} \oplus \mathcal{D}$ and $\mathcal{C} \ominus \mathcal{D}$, and more generally, any class of the form $\mathrm{Grid}(M)$, where M is a matrix of classes whose entries all belong to NLOL.*

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Corollary

For any sequence of classes $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k \in$ NLOL, the class $\mathcal{C}_1 \odot \mathcal{C}_2 \odot \cdots \odot \mathcal{C}_k$ is in NLOL, and therefore polynomially recognizable.

2D-NLOL-recognizable classes

Definition

Let $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$ be a sequence of distinct points in general position. We say that the sequence is top-right monotone if for every $i \in [k]$ the point (x_i, y_i) is to the right or above all the previous points of the sequence.

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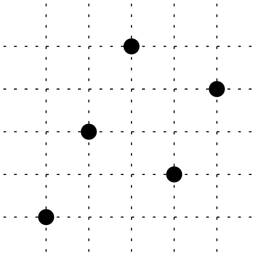

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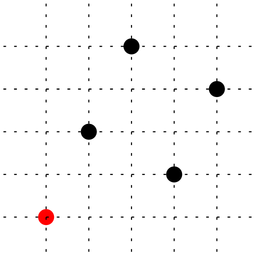

Possible sequence: $(1, 1)$

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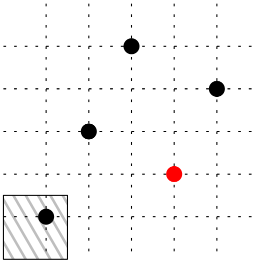

Possible sequence: $(1, 1)$, $(4, 2)$

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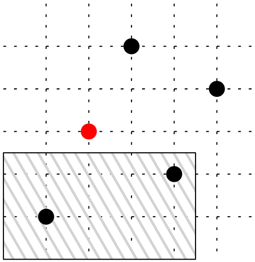

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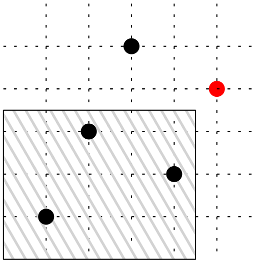

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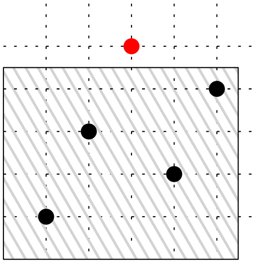

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Theorem

If $\mathcal{C}$ is a 2D-NLOL-recognizable class and $\mathcal{D}$ is any class of the set $\{Av(2413, 3142), Av(213), Av(231), Av(132), Av(312)\}$, then $\mathcal{C} \odot \mathcal{D}$ is polynomially recognizable.

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Proposition

The $(Av(\sigma) \odot Av(21))$-recognition problem is tractable whenever $\sigma = (1 \dots k) \oplus (l \dots 1) \oplus (1 \dots m)$ for $k \geq 0$, $l \geq 2$ and $m \geq 1$.

Summary

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- There is a large set NLOL of permutation classes such that for any $\mathcal{C}$, $\mathcal{D} \in$ NLOL the $(\mathcal{C} \odot \mathcal{D})$-recognition is tractable. Specific examples include layered permutations, co-layered permutations, merge of k increasing sequences, separable permutations with constant depth of their separating tree etc.

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- For slightly restricted set 2D-NLOL of permutation classes we can recognize its merges with "treelike" permutation classes that include separable permutations and Av(213) (plus all its symmetries).
- The problem becomes hard for forbidden patterns that are simple and of length at least 4.

Open questions

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Question

What is the complexity of $(\mathcal{C} \odot \mathcal{D})$-recognition when $\mathcal{C}$ and $\mathcal{D}$ are any two (possibly identical) classes from the set $\{\mathrm{Av}(2413, 3142), \mathrm{Av}(213), \mathrm{Av}(231), \mathrm{Av}(132), \mathrm{Av}(312)\}$?

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For which classes $\mathcal{C}$ is the $(\mathcal{C} \odot \text{Av}(21))$-recognition polynomial?

Thank you!