Enumeration of super-strong Wilf equivalence classes of permutations

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joint work with **Christina Savvidou**

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- **Generalized Factor Order**: Let $u, w \in \mathbb{P}^*$. Then $w \geq u$ if there is at least one embedding of u in w.
- **Weight generating function**:

$$F(u; t, x) = \sum_{w \geq u} wt(w)$$

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- Embedding generating function

$$A_u(x, y, z) = \sum_{w \in \mathbb{P}^*} x^{|w|} y^{||w||} z^{|\mathrm{Emb}(u,w)|}$$

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$u \sim_{ss} v \Leftrightarrow \exists$ a weight-preserving bijection $f : \mathbb{P}^* \to \mathbb{P}^*$ such that $\mathrm{Emb}(u,w) = \mathrm{Emb}(v,f(w))\ \forall w \in \mathbb{P}^*$.

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$$u \sim_{ss} v \Rightarrow u \sim_s v \Rightarrow u \sim v$$

Wilf equivalence

- Kitaev, Liese, Remmel, Sagan, *Rationality, irrationality, and Wilf equivalence in generalized factor order*, The Electronic Journal of Combinatorics **16** (2) #R22 (2009).

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- Hadjiloucas, M., Savvidou, *O*n super-strong Wilf equivalence classes of permutations, The Electronic Journal of Combinatorics, **25** (2) #P2.54 (2018).

Pyramidal sequence of consecutive differences

Definition

Let $u \in \mathcal{S}_n$ and $s = s_1 \cdots s_i \cdots s_n = u^{-1}$. Order the alphabet set of the suffix $s_i \cdots s_n$ of s from smallest to largest indices and define $\Delta_i(s)$ to be the vector of the corresponding *consecutive differences*. The sequence

$$p(s) = (\Delta_1(s), \Delta_2(s), \ldots, \Delta_{n-2}(s), \Delta_{n-1}(s))$$

has a pyramidal form and is called the pyramidal sequence of (consecutive) differences of $s \in \mathcal{S}_n$.

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$$4 < 5 < 6 < 7 < 8 \qquad \Delta_4(s) = (1, 1, 1, 1)$$

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\end{array}
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Theorem

Let $u, v \in \mathcal{S}_n$ and $s = u^{-1}, t = v^{-1}$. Then

$$u \sim_{ss} v \iff p(s) = p(t).$$

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$$\Delta_{i+1} = \begin{cases} (d_1, \ldots, d_{k-1}, d_k + d_{k+1}, d_{k+2}, \ldots, d_{n-i}), & \text{A} \quad \text{or} \\ (\not{d_1}\ d_2, \ldots, d_{n-i-1}, d_{n-i}), & \text{B} \quad \text{or} \\ (d_1, d_2, \ldots, d_{n-i-1}\ \not{d_{n-i}}). & \text{C} \end{cases}$$

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Note that if $\Delta_i = (\underbrace{d, d, \ldots, d}_{n-i})$, options B and C coincide.

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Let Π_n denote the set of all pyramidal sequences of the above form.

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Example

Construction of a pyramidal sequence

$$\Delta_1 = (1,1,1,1,1) \quad \longrightarrow \quad |\bullet|\bullet|\bullet|\bullet|\bullet|$$

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\Delta_3 = (1,1,3) &\longrightarrow |\bullet|\bullet|\bullet\ \ \bullet\ \ \bullet| \\
\Delta_4 = (1,3) &\longrightarrow |\bullet|\bullet\ \ \bullet\ \ \bullet|
\end{aligned}$$

Walls and balls representation

We view Δ_1 as a configuration of $\mathbf{n}$ **walls** which define $\mathbf{n-1}$ **chambers with one ball** in each. The transition from Δ_i to Δ_{i+1} is realized by a **removal of a wall**. If this wall is internal, we get Case A. If it is a left external (resp., a right external) one, we get Case B (resp., Case C).

Example

Construction of a pyramidal sequence

$$
\begin{array}{rcl}
\Delta_1 = (1,1,1,1,1) & \longrightarrow & |\bullet|\bullet|\bullet|\bullet|\bullet| \\
\Delta_2 = (1,1,2,1) & \longrightarrow & |\bullet|\bullet|\bullet\ \ \bullet|\bullet| \\
\Delta_3 = (1,1,3) & \longrightarrow & |\bullet|\bullet|\bullet\ \ \bullet\ \ \bullet| \\
\Delta_4 = (1,3) & \longrightarrow & |\bullet|\bullet\ \ \bullet\ \ \bullet| \\
\Delta_5 = (1) & \longrightarrow & |\bullet|
\end{array}
$$

Trapezoidal sequence of consecutive differences

For a fixed $i \in [n-2]$ a trapezoidal sequence of consecutive differences of height i is a sequence of the $i+1$ initial parts of a given pyramidal sequence such that $\Delta_{i+1} = (d, d, \ldots, d)$ and for each $j \in [2, i)$, $\Delta_j \neq (e, e, \ldots, e)$ for $e \in \mathbb{P}$. Denote the set of all such trapezoidal sequences by $\Delta_{i,n}$.

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Example

$$\Delta_8 = (4)$$
$$\Delta_7 = (4,\ 2)$$
$$\Delta_6 = (2,\ 2,\ 2)$$
$$\Delta_5 = (1,\ 2,\ 2,\ 2)$$
$$\Delta_4 = (1,\ 2,\ 2,\ 2,\ 1)$$
$$\Delta_3 = (1,\ 2,\ 1,\ 1,\ 2,\ 1)$$
$$\Delta_2 = (1,\ 1,\ 1,\ 1,\ 1,\ 2,\ 1)$$
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$$\Delta_3 = (1, 2, 1, 1, 2, 1)$$
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Non-interval permutations

Let $\mathcal{A}_n$ be the set of non-interval permutations, i.e., permutations of size $n \geq 2$ such that any proper prefix of length $l \geq 2$ is not, up to order, equal to an interval. Set $a_n = |\mathcal{A}_n|$.

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This is (A077607) in OEIS and is the convoluntory inverse of the factorial sequence, i.e., $(\sum_{k\geq 0} b_{k+1} t^k) \cdot (\sum_{k\geq 0} (k+1)! t^k) = 1$.

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$$\sum_{k=1}^{i} a_{k+1} \cdot (i - k + 1)! = (i + 1)!.$$

Prefixes of generalized non-interval permutations

A word of length l (resp., a set of cardinality l) is **periodic** if its vector of consecutive differences is equal to

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Definition

Let $\mathcal{D}_{i,n}$ be the set of all words $u = u_1 u_2 \ldots u_i$ of length i that appear as a non-empty prefix of a permutation w of size n whose remaining $(n-i)-$suffix is periodic and for all $j < i$ the set $[n] \setminus \{u_1, u_2, \ldots, u_j\}$ is not periodic.

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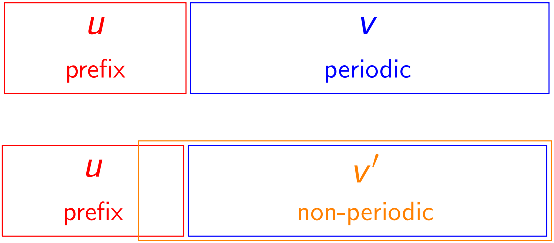
Set $d_{i,n} = |\mathcal{D}_{i,n}|$.

u	v
prefix	periodic


u
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u
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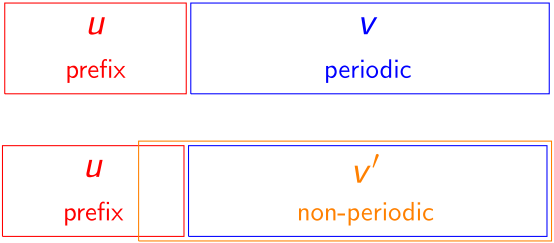

Example

The word 2 4 lies in $\mathcal{D}_{2,5}$ since it is a prefix of the permutation 2 4 1 3 5 and for the proper non-empty prefix 2 of 2 4 the remaining letters 1, 3, 4 and 5 can not constitute a periodic word.

$\mathcal{D}_{i,n} \longleftrightarrow \Delta_{i,n}$

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$$\Delta_5 = (1,2,2,2)$$

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$\Delta_1 = (1,1,1,1,1,1,1,1)$	$\rightarrow$	$\vert_1 \bullet \vert_2 \bullet \vert_3 \bullet \vert_4 \bullet \vert_5 \bullet \vert_6 \bullet \vert_7 \bullet \vert_8 \bullet \vert_9$		
$\Delta_2 = (1,1,1,1,1,2,1)$	$\rightarrow$	$\vert_1 \bullet \vert_2 \bullet \vert_3 \bullet \vert_4 \bullet \vert_5 \bullet \vert_6 \bullet\ 7 \bullet \vert_8 \bullet \vert_9$	$\rightarrow$	7
$\Delta_3 = (1,2,1,1,2,1)$	$\rightarrow$	$\vert_1 \bullet \vert_2 \bullet\ 3 \bullet \vert_4 \bullet \vert_5 \bullet \vert_6 \bullet\ 7 \bullet \vert_8 \bullet \vert_9$	$\rightarrow$	3
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$\Delta_5 = (1,2,2,2)$	$\rightarrow$	$\vert_1 \bullet \vert_2 \bullet\ 3 \bullet \vert_4 \bullet\ 5 \bullet \vert_6 \bullet\ 7 \bullet \vert_8\ \ 9$		

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$$\Delta_6 = (2,\ 2,\ 2)$$

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$\Delta_3 = (1,2,1,1,2,1)$	$\to$	$\vert_1 \bullet \vert_2 \bullet\ 3 \bullet \vert_4 \bullet \vert_5 \bullet \vert_6 \bullet\ 7 \bullet \vert_8 \bullet \vert_9$	$\to$	3
$\Delta_4 = (1,2,2,2,1)$	$\to$	$\vert_1 \bullet \vert_2 \bullet\ 3 \bullet \vert_4 \bullet\ 5 \bullet \vert_6 \bullet\ 7 \bullet \vert_8 \bullet \vert_9$	$\to$	5
$\Delta_5 = (1,2,2,2)$	$\to$	$\vert_1 \bullet \vert_2 \bullet\ 3 \bullet \vert_4 \bullet\ 5 \bullet \vert_6 \bullet\ 7 \bullet \vert_8\ \ 9$	$\to$	9
$\Delta_6 = (2,\ 2,\ 2)$	$\to$	$1\ \ \vert_2 \bullet\ 3 \bullet \vert_4 \bullet\ 5 \bullet \vert_6 \bullet\ 7 \bullet \vert_8\ \ 9$		

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$$\Delta_5 = (1,2,2,2) \quad \to \quad \vert_1 \bullet \vert_2 \bullet \; {}_3 \bullet \vert_4 \bullet \; {}_5 \bullet \vert_6 \bullet \; {}_7 \bullet \vert_8 \; {}_9 \quad \to \quad 9$$

$$\Delta_6 = (2,\ 2,\ 2) \quad \to \quad {}_1 \; \vert_2 \bullet \; {}_3 \bullet \vert_4 \bullet \; {}_5 \bullet \vert_6 \bullet \; {}_7 \bullet \vert_8 \; {}_9 \quad \to \quad 1$$

Super-strong Wilf equivalence classes

Theorem

Let s_n be the number of distinct super-strong Wilf equivalence classes of $\mathcal{S}_n$. Then

$$s_n = s_{n-1} + \sum_{i=2}^{n-2} d_{i,n} \cdot s_{n-i}.$$

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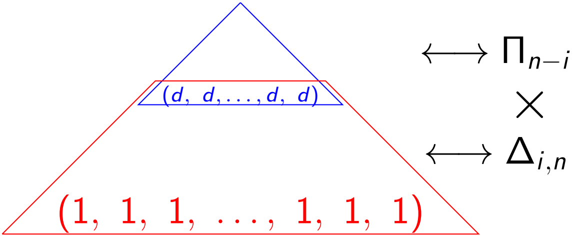

Tree representation T_5 of Π_5

$$(1, 1, 1, 1)$$

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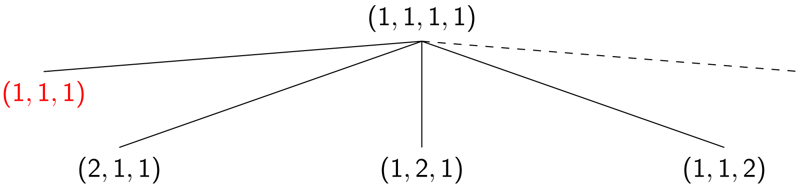

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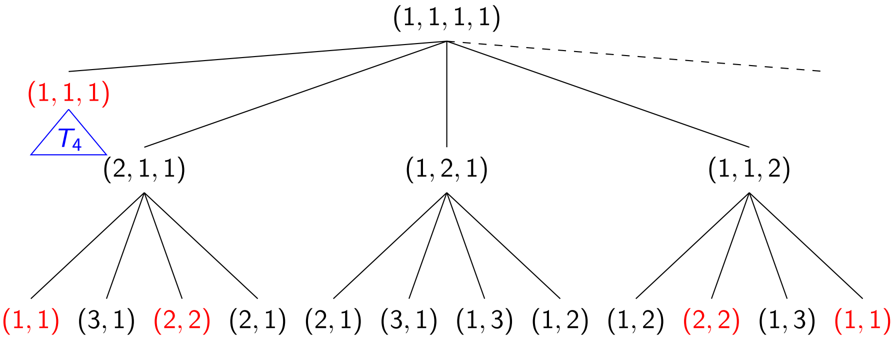

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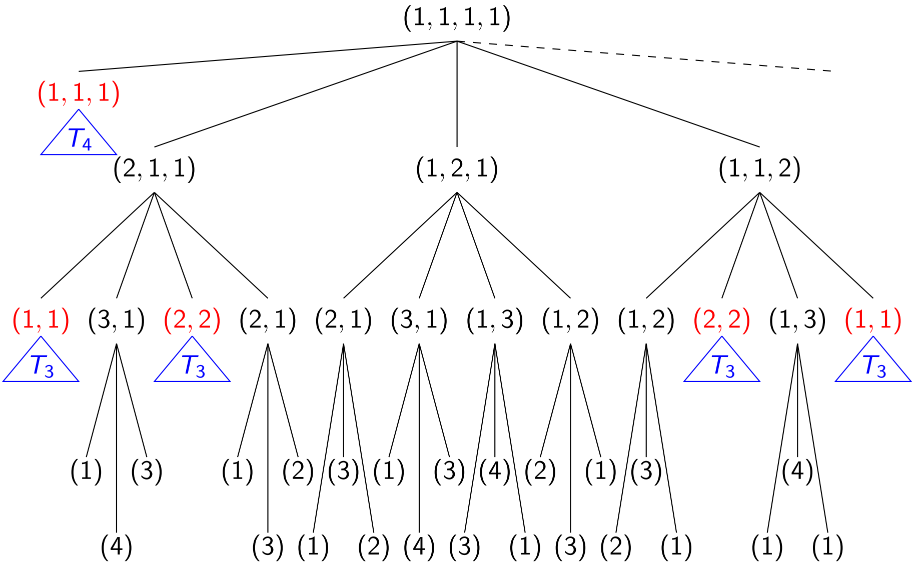

Super-strong Wilf equivalence classes with 2^j elements

Let $s_{j,n}$ be the number of super-strong Wilf equivalence classes of order 2^j in $\mathcal{S}_n$.

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Enumeration of $\mathcal{D}_{i,n}$

Let $q_{l,m}$ and $r_{l,m}$ be the unique quotient and remainder, respectively, of the Euclidean division of an arbitrary integer l with m.

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Let $n \geq 4$. For $i \in [n-2]$ and $m = n-i-1$, we have

$$\sum_{k=1}^{i} \frac{q_{n-k,m}}{2} \cdot (r_{n-k,m} + i - k + 1) \cdot d_{k,n} \cdot (i-k)! = \frac{q_{n,m}}{2} \cdot (r_{n,m} + i + 1) \cdot i!.$$

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Proposition

For $k < \lfloor \frac{n}{2} \rfloor$,

$$\mathcal{A}_{k+1} \longleftrightarrow \mathcal{D}_{k,n}.$$

List of super-strong Wilf equivalence classes

Let $red(v)$ be the reduced form of v, and set

$$\mathcal{E}_{i,n} = \begin{cases} \{1\}, & i = 1 \\ \mathcal{D}_{i,n}, & i \in [2, n-2]. \end{cases}$$

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Set

$$\mathcal{R}_n = \{ \color{red}{u} \cdot \color{blue}{v} \;:\; u \in \mathcal{E}_{i,n};\; red(v) \in \mathcal{R}_{n-i};\; i \in [n-2] \}.$$

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Theorem

A list of super-strong Wilf equivalence class representatives in $\mathcal{S}_n$ is given by the set

$$\mathcal{C}_n = \{w^{-1} : w \in \mathcal{R}_n\}.$$

Tree representation of $\mathcal{R}_5$

$\varnothing$

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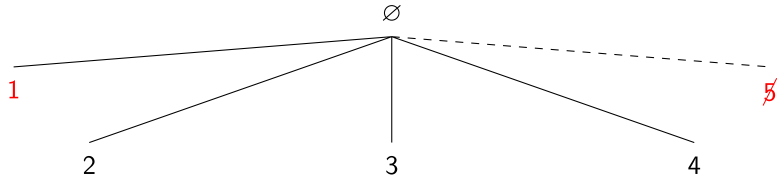
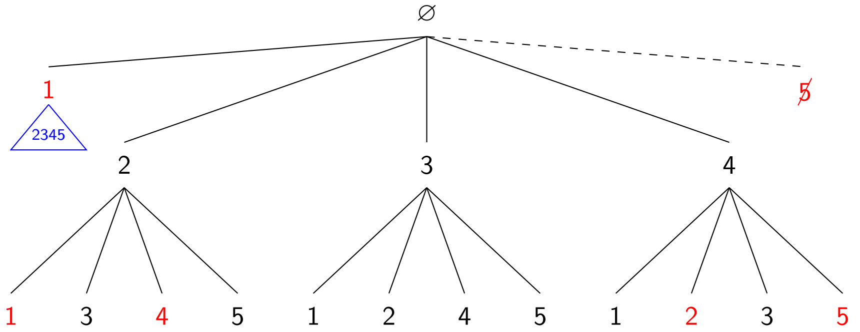


Tree representation of $\mathcal{R}_5$



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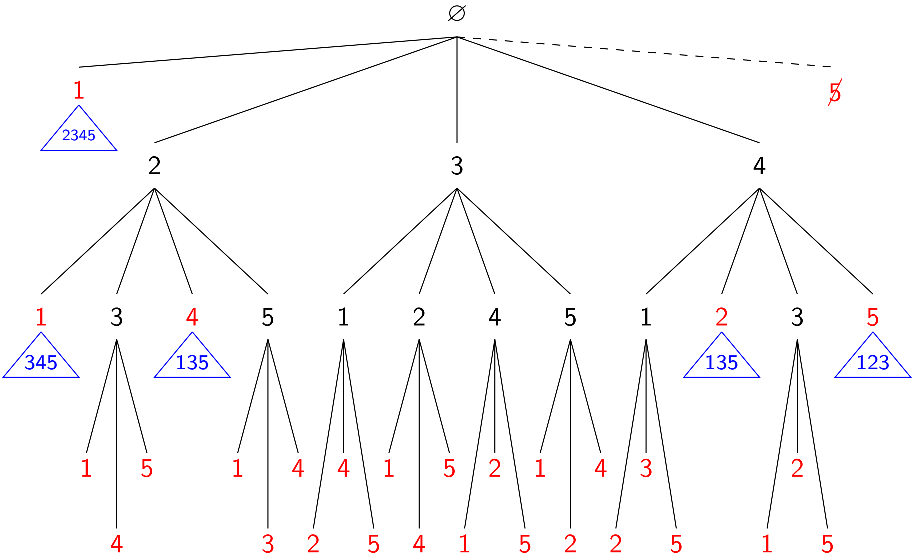

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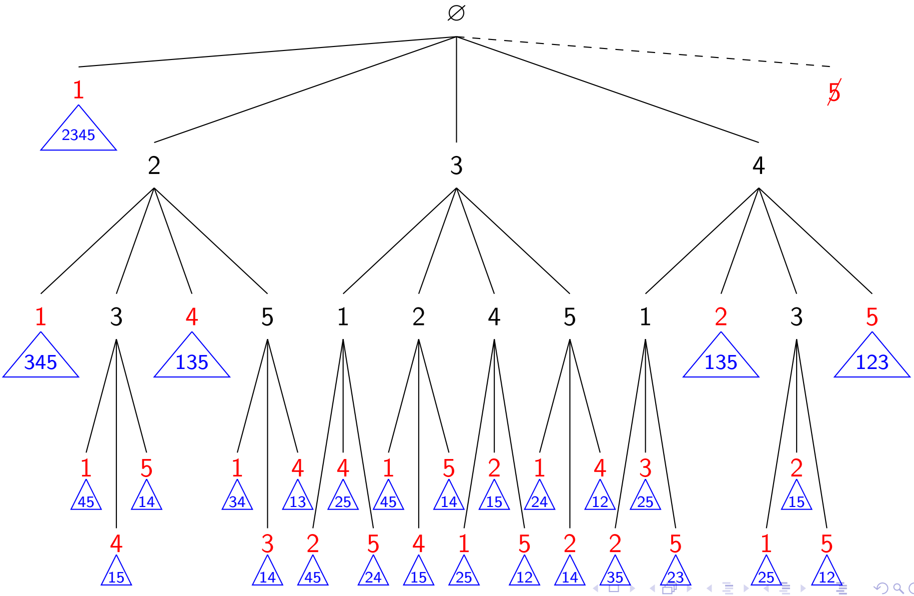

Shift Equivalence

J. Fidler, D. Glasscock, B. Miceli, J. Pantone, and M. Xu, *Shift equivalence in the generalized factor order*, Arch. Math. **110** (2018), 539-547.

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- The **skyline diagram** of a word $u = u_1 u_2 \ldots u_n \in \mathbb{P}^*$ is the geometric figure formed by adjoining n columns of squares such that the i-th column is made up of u_i squares.

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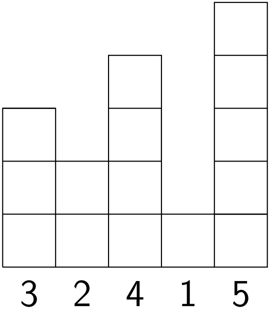
- Two words u and v are **shift equivalent** if v can be obtained by starting with u and performing any sequence of reversals and rigid shifts.

Rigid Shifts



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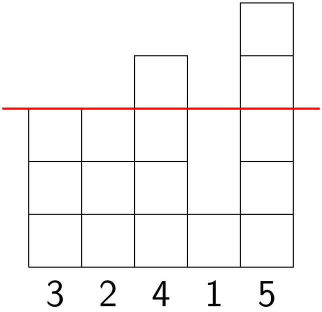

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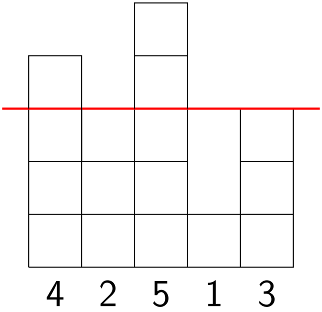

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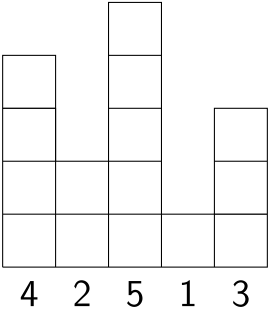

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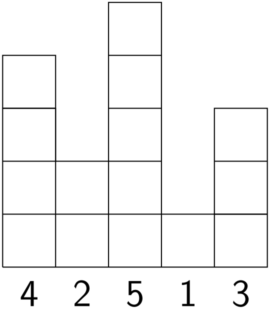

Definition

Two permutations are **strongly shift equivalent** if one can be obtained from the other by performing any sequence of rigid shifts.

Enumeration of Shift Equivalence Classes

Theorem

Let $u, v \in \mathcal{S}_n$. Then u is strongly shift equivalent to v if and only if $u \sim_{ss} v$.

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$$\begin{array}{l} \Delta_{i+1} = (d, d, \ldots, d) \\ \quad\quad\quad\quad \downarrow \\ \Delta_i = (d, \quad d, \quad \ldots, \quad d) \end{array} \quad \leftrightarrow \quad \text{rigid shift of height } i \text{ by } d \text{ places}$$

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Corollary

Let sh_n be the number of shift equivalence classes in $\mathcal{S}_n$. For $n \geq 3$, we have

$$sh_n = 2 + \frac{s_n - 2}{2}.$$

The numbers $d_{i,n}$ and s_n

$i \backslash n$	3	4	5	6	7	8	9	10	11	12
1	3	2	2	2	2	2	2	2	2	2
2		6	4	2	2	2	2	2	2	2
3			24	16	14	8	8	8	8	8
4				168	100	80	68	44	44	44
5					1,212	712	500	488	416	296
6						10,824	6,376	4,664	3,704	3,512
7							103,992	58,336	43,592	33,152
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Table 1: The numbers $d_{i,n}$ for $1 \leq i \leq 10$ and $3 \leq n \leq 12$

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n	1	2	3	4	5	6	7	8	9	10	11	12
s_n	1	1	2	8	40	256	1,860	15,580	144,812	1,490,564	16,758,972	205,029,338

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- M. and Savvidou, *Enumeration of super-strong Wilf equivalence classes of permutations*, arXiv:1803.08818 [math.CO]

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Further Research

- Generating functions for $d_{i,n}$, s_n, and sh_n.

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- Generating functions for $d_{i,n}$, s_n, and sh_n.
- Analogous results connecting super strong Wilf equivalence with strong Wilf equivalence, Wilf equivalence and other types of such equivalences.

Thank you!