A, B-Minimal Stirling Numbers

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Outline

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1. The Basics
2. A, B-Minimal Stirling Numbers

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2. A, B-Minimal Stirling Numbers
3. $[r], \{r+1\}$-Minimal Stirling Numbers

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Definitions

Set partitions

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$$min(69/4/38/1257) = \{1, 3, 4, 6\}.$$

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Let $\left\{ {n \atop k} \right\}$ denote the *Stirling number of the second kind*, defined by the recursive relationship

$$\left\{ {n+1 \atop k} \right\} = \left\{ {n \atop k-1} \right\} + k \left\{ {n \atop k} \right\},$$

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Theorem

For all $n, k \in \mathbb{Z}$, $\left\{ {n \atop k} \right\} = |\Pi_{n,k}|$.

Counting Other Collections of Set Partitions

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1. For any set partition π enumerated by $\left\{ {n \atop k} \right\}_r$, we must have $[r] \subseteq min(\pi)$.
2. This does not exclude $r' > r$ from being a minimal element of a part.

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From looking at the recursion, we find that these count certain m-tuples of set partitions, with the restriction that if π, π' are elements of the same tuple, then $min(\pi) = min(\pi')$.

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The two generalizations we illustrated implicitly force some structure on the minimal elements of our parts.

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We call $\mathcal{B}_n(A;B)$ the *n-th A,B-minimal Bell number of the second kind*.

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We now consider two special cases of minimal A, B-Stirling numbers.

$[r], \{r+1\}$-Minimal Stirling Numbers

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For $1 \leq r < n$, $S_{n,[r]} = r \sum_{i=0}^{n-r-1} \binom{n-r-1}{i} r^i \mathcal{B}_{n-r-1-i}$.

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Proof.

Each partition counted by $S_{n,[r]}$ contains $r+1$ in one of the first r parts, and some $0 \leq i \leq n-r-1$ of the elements of $\{r+2, \ldots, n\}$ in the first r parts as well, giving the term $r \cdot \binom{n-r-1}{i} \cdot r^i$ for any fixed i.

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A couple of things to think about

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$$[[S_{n,[r]}]]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & -4 & 1 & 0 & 0 & 0 \\ -3 & 8 & 9 & 0 & -5 & 1 & 0 & 0 \\ -17 & -8 & 27 & 16 & 0 & -6 & 1 & 0 \\ -33 & -136 & 9 & 64 & 25 & 0 & -7 & 1 \end{bmatrix}$$

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Theorem

For a positive integer n, $\displaystyle\sum_{k=1}^{n} \tilde{S}_{n,\{k\}} = \mathcal{B}_{n+1} - \mathcal{B}_n$.

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We say that σ *avoids* the pattern 1-32 if there do not exist $1 \leq i < j < n$ such that $\sigma_i < \sigma_{j+1} < \sigma_j$. We define

$$Av_n^{\uparrow}(1\text{-}32) = \{\sigma \in \mathcal{S}_n \mid \sigma \text{ avoids 1-32 and } \sigma_{n-1} < \sigma_n\}.$$

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Corollary

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For example, if $\sigma = 968524137$, then $S_1 = 9$, $S_2 = 68$, $S_3 = 5$, $S_4 = 24$, and $S_5 = 137$.

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For postive n, $\tilde{S}_{n,\{i\}} = \left|\{\sigma \in Av^{\uparrow}_{n+1}(1\text{-}32) \mid \sigma_{n+1} = n+2-i\}\right|.$

$\{i\}$-Minimal Stirling Numbers

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The End

Thanks for listening!

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Questions?

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