The principal Möbius function of permutations with opposing adjacencies

David Marchant
Joint work with Robert Brignall

Permutation Patterns 2018
Dartmouth College, Hanover, NH.

13th July 2018

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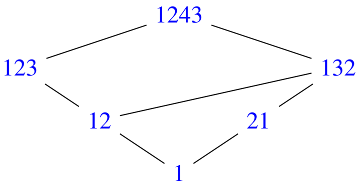

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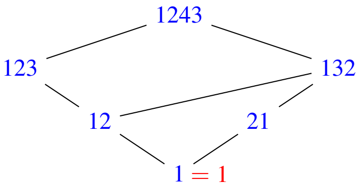

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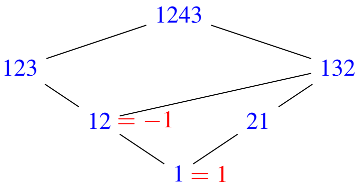

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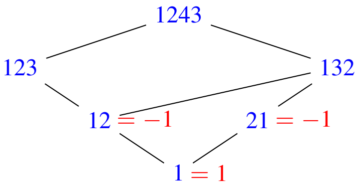

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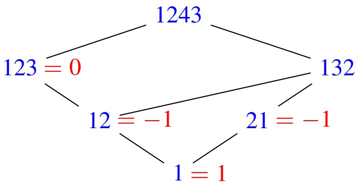

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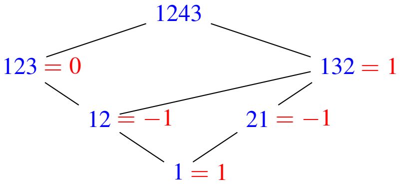

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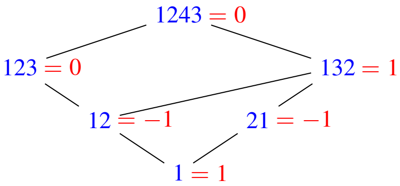

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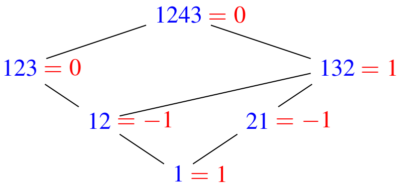
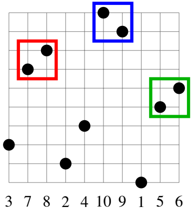


$$\text{If } \pi \neq 1, \text{ then } \sum_{[1,\pi]} \mu[\lambda] = 0.$$

. . . and what are opposing adjacencies?



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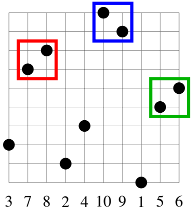

We have opposing adjacencies when we have an interval isomorphic to 12 (an *up-adjacency*), and an interval isomorphic to 21 (a *down-adjacency*).

Opposing adjacencies

Theorem (Brignall et al. 2018+)

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- Choose an up-adjacency and a down-adjacency.
- Construct three permutations:
  - λ, where we replace the left adjacency (of the two chosen) with a single point.
  - ρ, where we replace the right adjacency with a single point.
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- Use these permutations to split the poset into five (overlapping) subsets, then use inclusion-exclusion.

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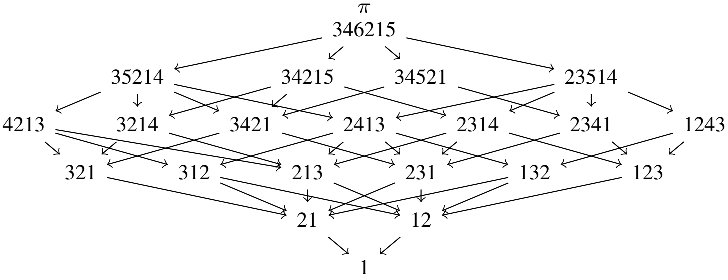

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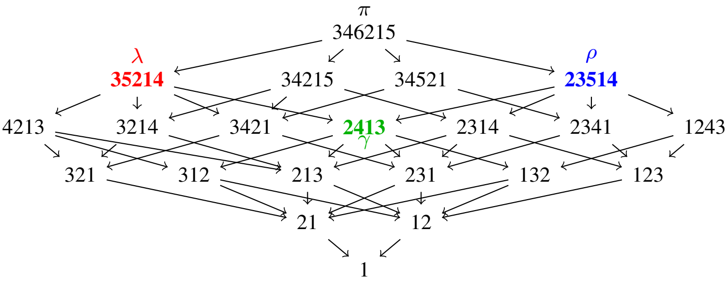

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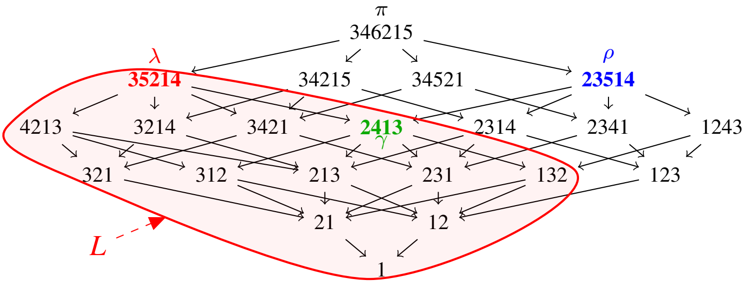

$L = [1, \lambda]$

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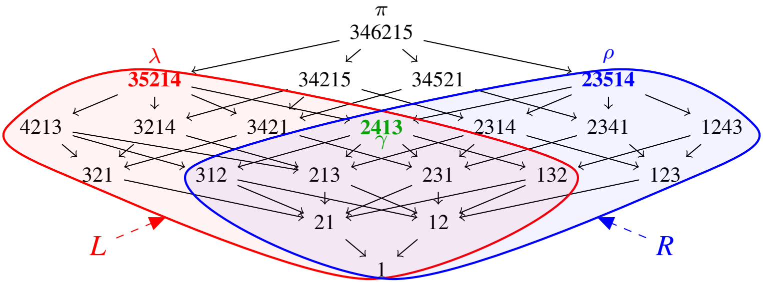

$L = [1, \lambda], \quad R = [1, \rho]$

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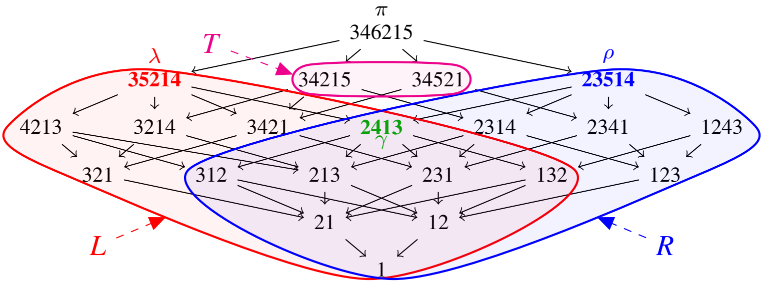

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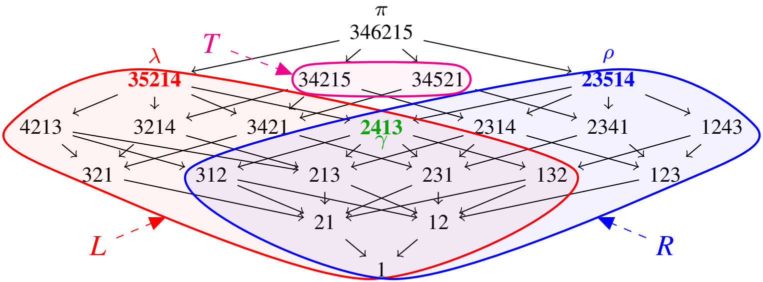

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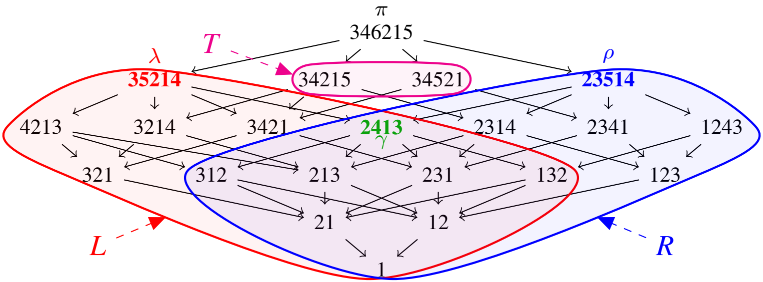

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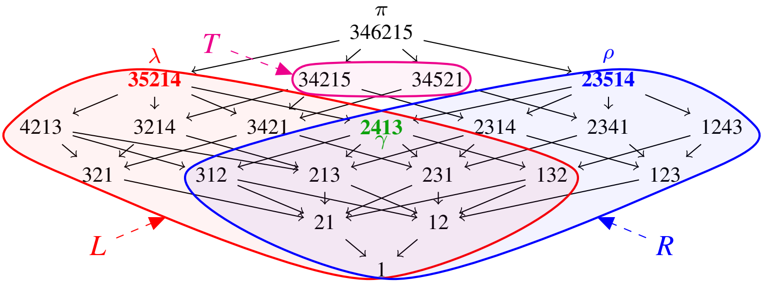

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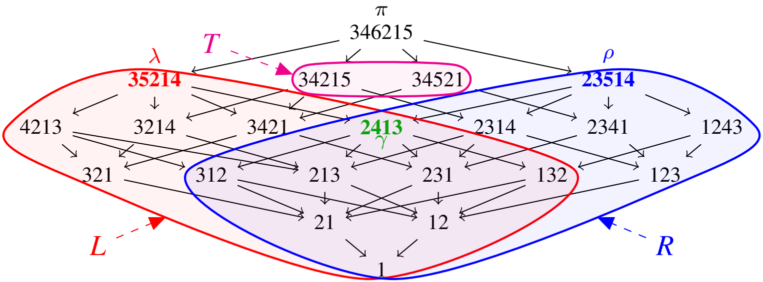

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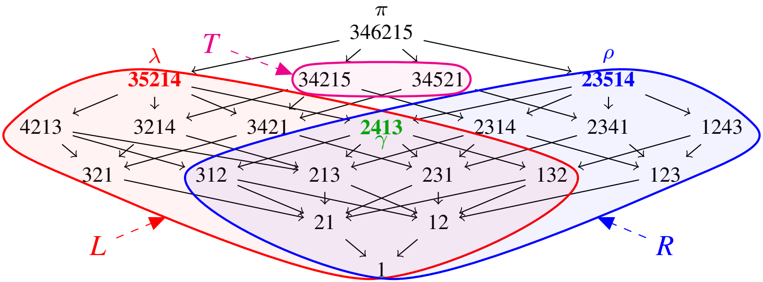

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- $L = [1, \lambda]$, $R = [1, \rho]$, and $G = [1, \gamma]$ are closed intervals.
- Every permutation in $T = [1, \pi) \setminus (L \cup R)$ or $X = (L \cap R) \setminus G$ has an opposing adjacency.

Proportion of permutations where $\mu[\pi] = 0$

Theorem (Brignall et al. 2018+)

The proportion of permutations of length n where the principal Möbius function is zero, $Z(n)$, is, asymptotically, bounded below by 0.3995.

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- As $n \to \infty$

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The Open University

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Figures were independently calculated by Jason Smith and by M.

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Conjecture

The proportion of permutations that have principal Möbius function value equal to zero is bounded above by $Z(10) \approxeq 0.6040$.

Figures were independently calculated by Jason Smith and by M.

Future directions

- An upper bound for $Z(n)$.
- Does a limit exist for $Z(n)$?
- And if a limit does exist, what is it?
- What about permutations that have multiple non-opposing adjacencies?
- We know that some permutations with multiple non-opposing adjacencies have a non-zero principal Möbius function value, so we need a criteria to exclude these.

Finally…

Thank you!

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