Enumerative and Algebraic Combinatorics of OEIS A071356

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Integer Sequences

- The Catalan numbers (A000108): $1, 1, 2, 5, 14, 42, \ldots$
- $\displaystyle\sum_{n=0}^{\infty} c_n x^n = 1 + x + 2x^2 + 5x^3 + 14x^4 + \cdots = \frac{1 - \sqrt{1 - 4x}}{2x}$
- The sequence A071356 is: $1, 1, 2, 6, 20, 72, \ldots$
- $\displaystyle\sum_{n=0}^{\infty} a_n x^n = 1 + x + 2x^2 + 6x^3 + 20x^4 + \cdots = \frac{2x + 1 - \sqrt{1 - 4x - 4x^2}}{4x}$

Inversion Sequences

Definition

The set of inversion sequences is $I_n = \{(e_1, \cdots, e_n) \in \mathbb{N}^n \mid e_i \leq i - 1\}$

- A bijection between inversion sequences and permutations is to use the Lehmer code where we define

$$e_i = |\{j \mid j < i \text{ and } \pi_j < \pi_i\}|$$

and then reverse the sequence.

Weakly Increasing Inversion Sequences

Definition

Let

$$C_n = \{(e_1, \ldots, e_n) \in I_n \mid e_1 \leq \ldots \leq e_n\}$$

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Theorem

The reverse Lehmer code bijection restricts to a bijection between elements of $S_n(\underline{13}2)$ and weakly increasing inversion sequences.

Corollary

$$|C_n| = c_n$$

Pattern Avoiding Inversion Sequences

Definition

We recall from [Martinez-Savage 2017] that elements of $\mathbf{I}_n(e_i > e_j \leq e_k)$ take the following form:

$$e_1 \leq \cdots \leq e_t > e_{t+1} > \cdots > e_n$$

for some t such that $1 < t \leq n$. Let t be called the *peak* of such an inversion sequence.

Question (Martinez-Savage 2017)

Is $|\mathbf{I}_n(e_i > e_j \leq e_k)|$ counted by OEIS A071356?

Theorem (H. 2018)

The generating function of $\mathbf{I}_n^R(e_i \geq e_j < e_k)$ *is*

$$\frac{2x + 1 - \sqrt{1 - 4x - 4x^2}}{4x}.$$

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- Given an inversion sequence of length n and peak t, there are two ways to break the sequence into two pieces so that the left piece is weakly increasing and the right piece is strictly decreasing.

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$$I_{s,k} = \{((e_1, \ldots, e_s), (e_{s+1}, \ldots, e_{s+k})) \mid (e_1, \ldots, e_s) \in C_s,\ s \geq e_{s+1} > \cdots\}$$

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We can count the size of $I_{s,k}$, by noting that the left piece is counted by a Catalan number, and the right piece is counted by a binomial coefficient.

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$$|I_{s,k}| = c_s \binom{s+1}{k}$$

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$$k = n - s$$

$$2\sum_{n=2}^{\infty} a_n x^n = \sum_{s=2}^{\infty} c_s x^s \sum_{k=0}^{\infty} \binom{s+1}{k} x^k + 2x^2 + x^3$$

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Proof continued.

By the binomial theorem:

$$2\sum_{n=2}^{\infty} a_n x^n = \sum_{s=2}^{\infty} c_s x^s (x+1)^{s+1} + 2x^2 + x^3$$

We note that by convention, $a_0 = a_1 = 1$.

$$2(A(x) - 1 - x) = \sum_{s=2}^{\infty} c_s x^s (x+1)^{s+1} + 2x^2 + x^3$$

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Proof continued.

We recall that the Catalan numbers have generating function $\sum_{s=0}^{\infty} c_s y^s = \frac{1 - \sqrt{1 - 4y}}{2y}$. Using the Catalan generating function for $y = x(1 + x)$, and after a routine computation, we find the desired generating function:

$$\sum_{n=0}^{\infty} a_n x^n = \frac{2x + 1 - \sqrt{1 - 4x(x + 1)}}{4x}$$

□

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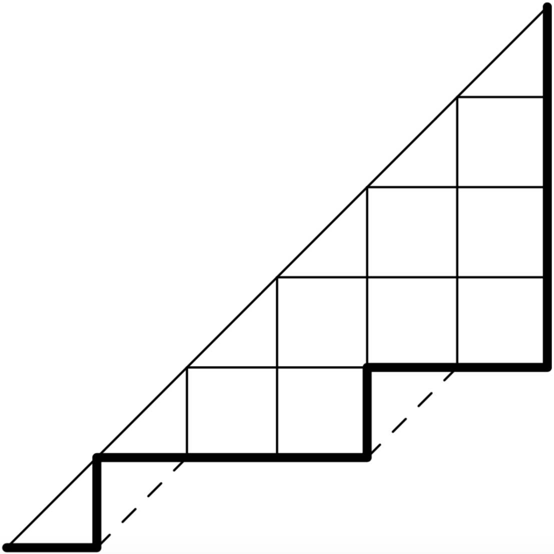
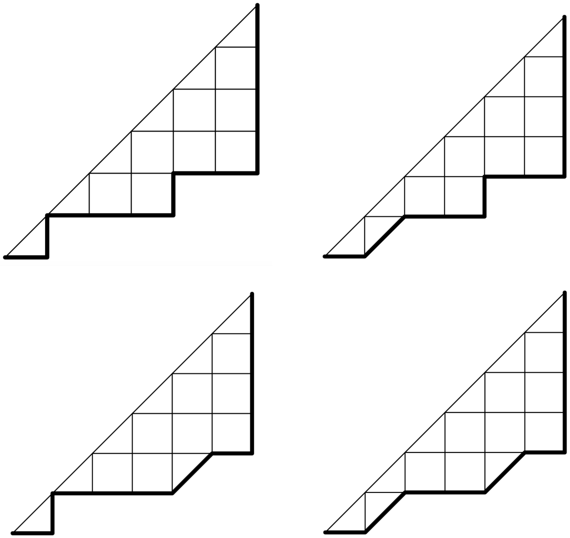
$$\mathcal{D}_n \subseteq \mathcal{RSP}_n \subseteq \mathcal{SP}_n$$

Building Schröder paths from Dyck paths



Lattice Path Examples



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- [Aguiar-Moreira 2006] showed that a certain family of trees is counted by OEIS A071356.
- The paths are in bijection with the trees.

Pattern Avoiding Permutations

Definition

$S_n(\underline{41}23, \underline{41}32, 2\underline{41}3, 3\underline{41}2)$ is the set of permutations such that for any descent $\pi_i\pi_{i+1}$ where $\pi_{i+1} - \pi_i \geq 3$, all the values between π_{i+1} and π_i occur to the left of π_i.

Theorem (H. 2015)

$|S_n(\underline{41}23, \underline{41}32, 2\underline{41}3, 3\underline{41}2)|$ *is counted by OEIS A071356.*

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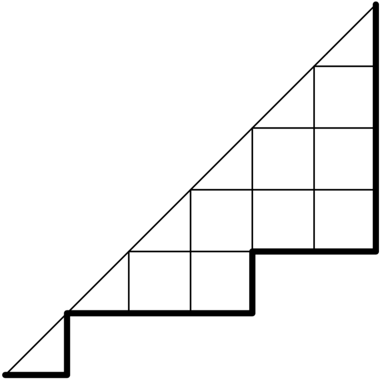
A *non-Dyck inversion* of a permutation $w \in S_n$ is an inversion (σ_i, σ_j) such that there exists some σ_k where $i < j < k$ and $\sigma_j < \sigma_k < \sigma_i$. A *Dyck inversion* of a permutation $\sigma \in S_n$ is an inversion (σ_i, σ_j) that is not a non-Dyck inversion.

τ example

$$\tau(641532) =$$



Using Dyck Inversions to define a map $\tau : S_n \to \mathcal{D}_n$

Definition

Let $\tau : S_n \to \mathcal{D}_n$ be the following map. Let

$$d_i = |\{j \mid (\sigma_i, \sigma_j) \text{ is a Dyck inversion}\}|$$

$\tau(\sigma)$ is the unique Dyck path where the east step in the ith column occurs at height $i - d_i + 1$.

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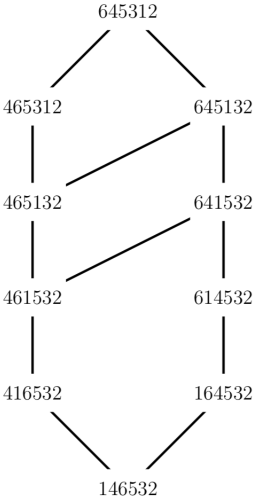
Theorem (Bandlow-Killpatrick 2001)

The map τ_{Av} is a bijection. Moreover, it is statistic preserving sending inversions to area.

τ-fibers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645132
465132
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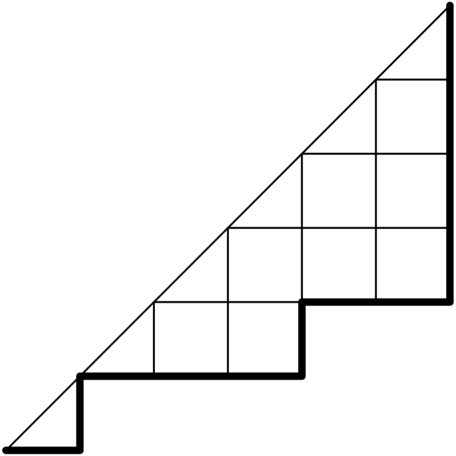
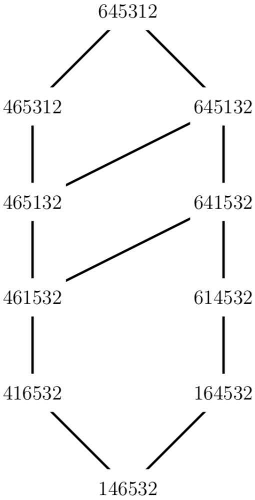
461532
614532
416532
164532
146532



τ-posets


645312
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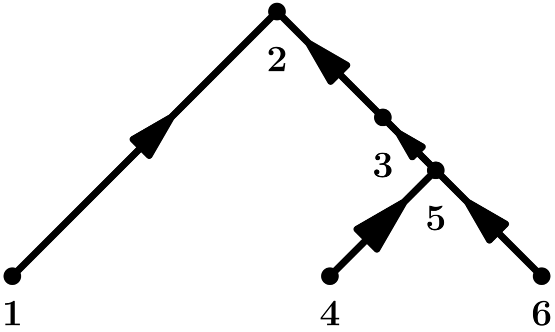
2
3
5
1
4
6

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- The top elements of the intervals have reverse Lehmer codes that are weakly increasing sequences.
- τ restricted to the top elements gives a bijection between the weakly increasing sequences and $\mathcal{D}_n$.

A map $\omega : S_n \to \mathcal{RSP}_n$

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- Given a permutation σ, build the Dyck path $\tau(\sigma)$.

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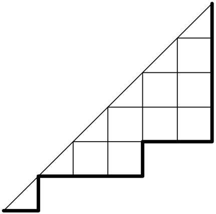
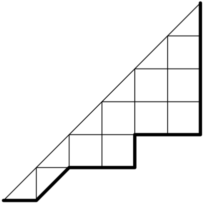
- Use the relative order of the atoms of the binary tree in the permutation to decide where the triangles appear.

ω examples

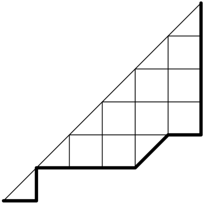
$\omega(146532)=$



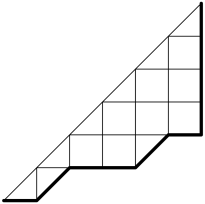
$\omega(164532)=$



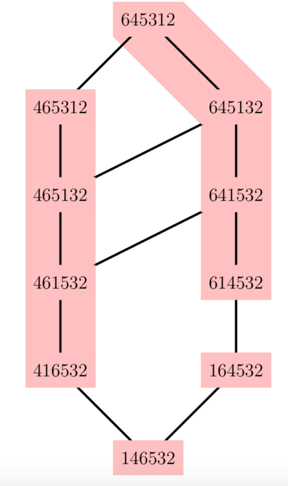
$\omega(465132)=$

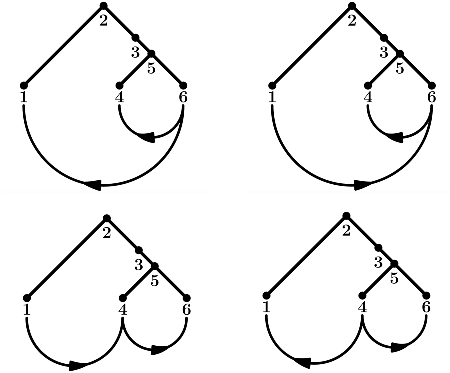


$\omega(641532)=$

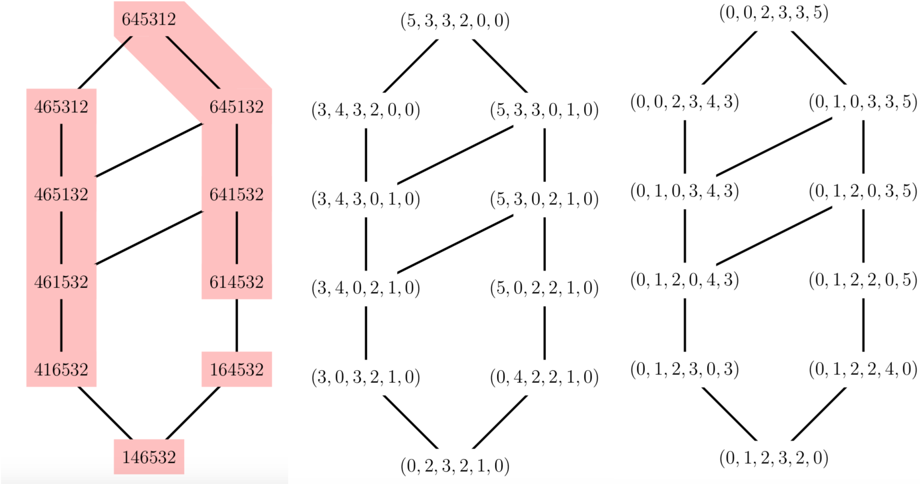


ω-fibers





ω-fiber codes



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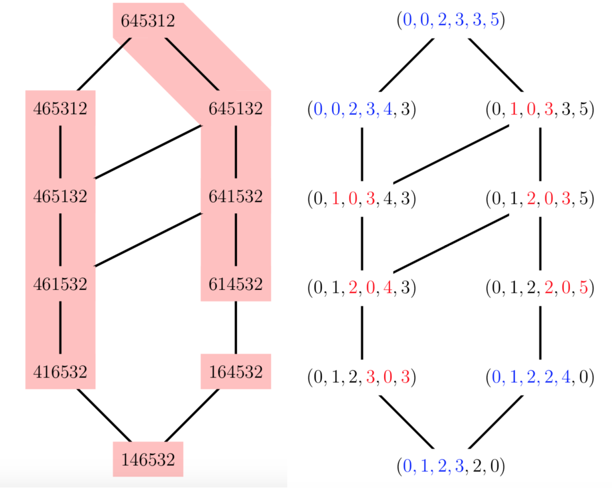

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- The Lehmer codes of the top elements of the intervals are precisely the pattern avoiding inversion sequences.
- The number of inversion sequences is the same as the number of paths.
- Therefore, ω restricted to the top elements gives the desired bijection.

Thank you for listening!