Prolific Permutations and Expected Breadth

Cheyne Homberger

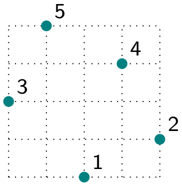
Permutation Patterns 2018

Joint work with Simon Blackburn and Pete Winkler

Definition

If π is a permutation of length n, then the **plot** of π is the set of points

$$\{(1,\pi(1)),(2,\pi(2)),\cdots(n,\pi(n))\}\subset\mathbb{R}^2$$



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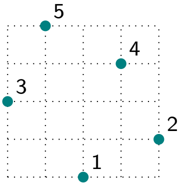
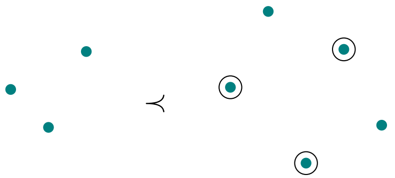


$\pi = 35142$

Permutation Patterns

Definition

Let $\pi = \pi(1)\pi(2)\cdots\pi(n)$ and $\sigma = \sigma(1)\sigma(2)\cdots\sigma(k)$ be two permutations. π **contains** σ **as a pattern** (written $\sigma \prec \pi$) if there is some subsequence $\pi(i_1)\pi(i_2)\ldots\pi(i_k)$ which is order isomorphic to the entries of σ (i.e., $\pi(i_j) < \pi(i_k)$ if and only if $\sigma(j) < \sigma(k)$).



$$213 \quad \prec \quad \mathbf{3}5\mathbf{14}2$$

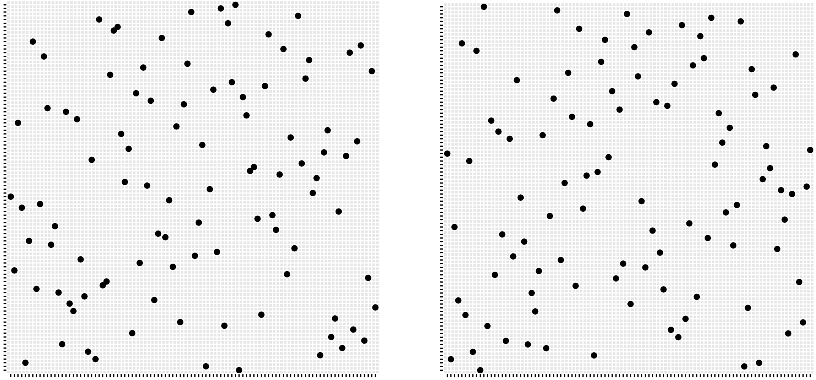
Permutation Breadth

The **breadth** of a permutation π is

$$\min_{i \neq j} \Big\{ |\pi(i) - \pi(j)| + |i - j| \Big\}.$$

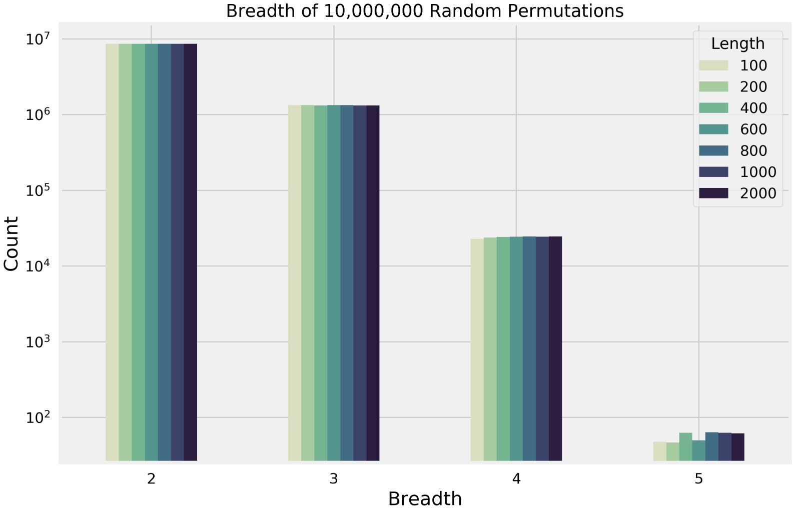
Permutation Breadth

The **breadth** of a permutation is the minimum pairwise manhattan distance between entries of its plot.

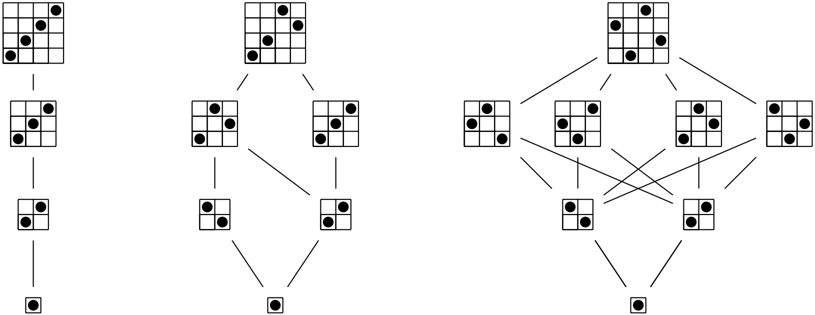


Breadth 2 Breadth 4

Random Permutations



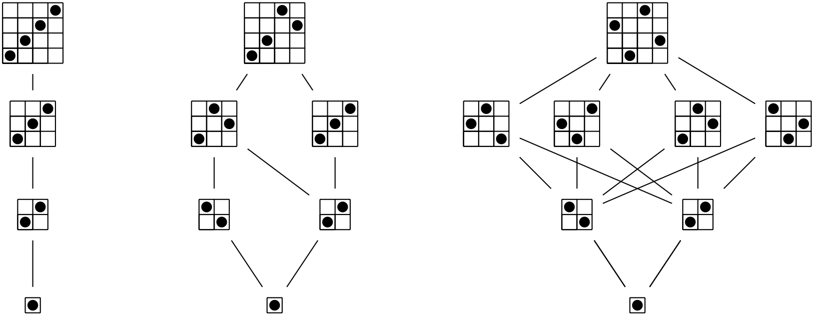
Principal Downsets and Prolific Permutations



Principal Downsets and Prolific Permutations

Definition

A permutation π of length n is d-**prolific** if each $n-d$ subset of entries forms a unique pattern. That is, the principal downset generated by π is as $\binom{n}{d}$-wide at rank $n-d$.



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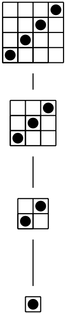
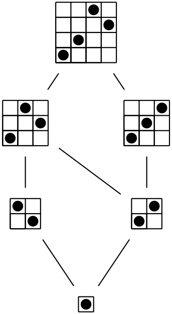


not prolific



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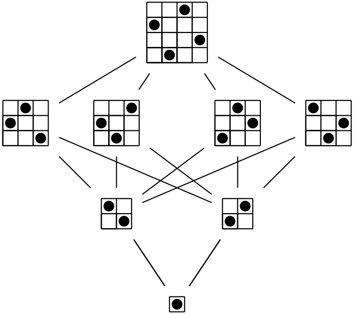

1-prolific

Characterizing Prolificity

Theorem (Bevan, H., Tenner)

A permutation is d-prolific if and only if its has breadth $\geq d + 2$.

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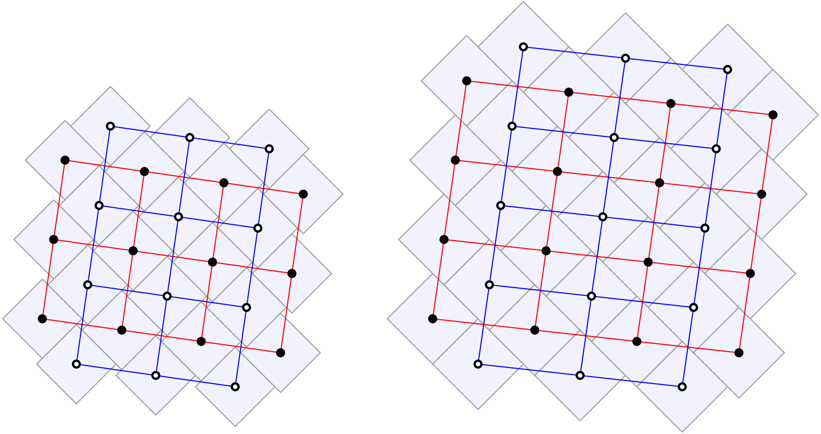
There exists an n-permutation with breadth $\geq d+2$ if and only if

$$n \geq d^2/2 + 2d + 1.$$

Maximum Breadth, Minimum Length



5-prolific 24-permutation and 6-prolific 31-permutation.

Question

How common are prolific permutations?

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Theorem (Blackburn, H., Winkler)

As $n \to \infty$, a random n-permutation has breadth $\geq d+2$, (and hence is d-prolific) with probability approaching

$$\mathbb{P}\left[\mathbf{br} \geq d+2\right] \to e^{-d^2-d}.$$

The Prolific Portion of Permutations

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Theorem (Blackburn, H., Winkler)

The expected breadth of a random permutation approaches

$$\mathbb{E}\left[\mathbf{br}\right] \to 1 + \sum_{d \geq 0} e^{-d^2-d} \approx 2.13782018\ldots.$$

Idea: Close Pairs

Fix d, n.

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$$|\pi(i) - \pi(j)| + |i - j| < d + 2.$$

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Note

A permutation has breadth $\geq d + 2$ (and hence is d-prolific) iff it has no close pairs.

Idea of Proof

Definition

For $1 \leq i \leq n$, let X_i be the indicator random variable for the pair i, j being a close pair for some $j > i$.

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Key Idea

Most X_i are **mostly** independent, $\sum_i X_i$ is **close** to the number of close pairs, and for **most** i, we have

$$\mathbb{P}\left[X_i\right] \approx \lambda / n$$

with $\lambda := 2\binom{d+1}{2} = d^2 + d$.

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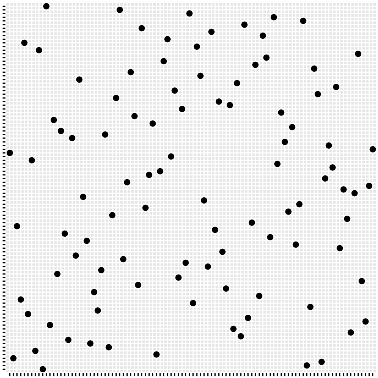
Fact (Poisson Approximation)

If we have n independent and identically distributed Bernoulli random variables each with mean λ / n, then, as $n \to \infty$, their sum is Poisson-λ.

Example



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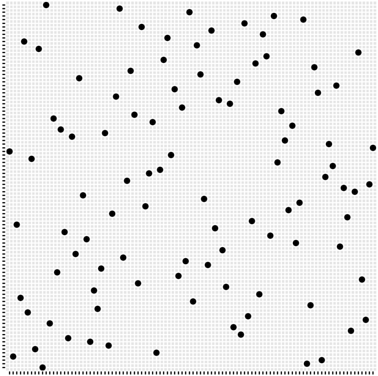
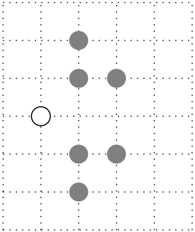
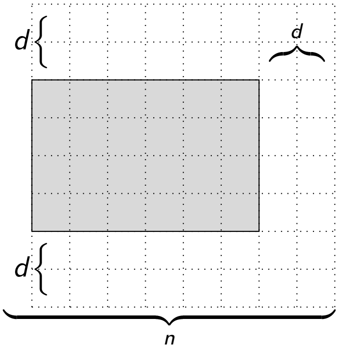




$\mathbb{P}[X_i = 1]$



Shaded region has $\sim n^2$ entries, while remainder has $\sim n$.

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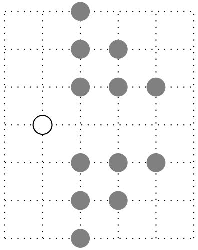

The filled dots mark the $2\binom{4}{2} = 3^2 + 3 = 12$ potential entries which will lead to the hollow dot starting a close pair.

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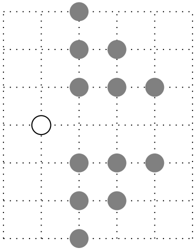

The filled dots mark the $2\binom{4}{2} = 3^2 + 3 = 12$ potential entries which will lead to the hollow dot starting a close pair.

In general we have $\lambda := 2\binom{d+1}{2} = d^2 + d$ potential spots which can make i into the start of a close pair.

Theorem

$$\mathbb{P}[X_i = 1] = \frac{\lambda}{n} + O(n^{-2}).$$

Rough Sketch

Strategy (Wolfowitz, 1944)

Let $X = \sum_{i=1}^{n} X_i$. Let Y_i be iid Bernoulli random variables with mean λ, and let $Y := \sum_{i=1}^{n} Y_i$.

We will show that

$$\lim_{n\to\infty} \mathbb{E}\left[X^t\right] = \lim_{n\to\infty} \mathbb{E}\left[Y^t\right].$$

Then, since $Y := \sum_{i=1}^{n} Y_i$ is asymptotically Poisson, we have that X is Poisson with mean λ, which completes our proof.

Distribution of Close Pairs

Theorem

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Corollary

The expected number of permutations with no d-close pairs is

$$e^{-\lambda} = e^{-d^2-d}.$$

Expected Breadth

Caveat

Knowing the asymptotic distribution of close pairs for a given d is **not strong enough** to calculate the expected breadth of a random permutation.

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Expectation

Let **br** be the breadth of the random permutation π. Then

$$\begin{aligned}\mathbb{E}[\mathbf{br}] &= 2 \cdot \mathbb{P}[\mathbf{br} = 2] + 3 \cdot \mathbb{P}[\mathbf{br} = 3] + 4 \cdot \mathbb{P}[\mathbf{br} = 4] + \cdots \\ &= 1 + \left(\underbrace{\mathbb{P}[\mathbf{br} \geq 2]}_{\to e^{0}} + \underbrace{\mathbb{P}[\mathbf{br} \geq 3]}_{\to e^{-2}} + \underbrace{\mathbb{P}[\mathbf{br} \geq 4]}_{\to e^{-6}} + \cdots \right)\end{aligned}$$

Boole's Inequality and the Bonferroni Inequalities

Let $\{A_i\}_{i=1}^n$ be events. Then

$$\mathbb{P}\left[\cup_i A_i\right] \leq \sum_i \mathbb{P}\left[A_i\right].$$

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Also, letting

$$S_1 := \sum_i \mathbb{P}\left[A_i\right], \;\; S_2 := \sum_{i<j} \mathbb{P}\left[A_i \cap A_j\right],$$

$$S_k := \sum_{i_1<i_2<\cdots<i_k} \mathbb{P}\left[A_{i_1} \cap \cdots \cap A_{i_k}\right],$$

we have, for all k,

$$\sum_{j=1}^{2k} (-1)^{j-1} S_j \leq \mathbb{P}\left[\bigcup_{i=1}^n A_i\right] \leq \sum_{j=1}^{2k+1} (-1)^{j-1} S_j.$$

Applying the Bonferroni Inequalities

Let X_i be the indicator variable for $\pi(i)$ being the initial entry in a close pair. Then

$$\mathbb{P}\left[\bigcup_i \{X_i = 1\}\right]$$

is the probability that π has breadth $\geq d$.
Therefore

$$\sum_{j=1}^{2k} (-1)^{j-1} S_j \leq \mathbb{P}\left[\mathbf{br} \geq d\right] \leq \sum_{j=1}^{2k+1} (-1)^{j-1} S_j,$$

where

$$S_k = \sum_{i_1 < \dots i_k} \mathbb{E}\left[X_{i_1} \cdots X_{i_k}\right].$$

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Lemma

Let d be a function of n such that $d = O(\log n)$. The probability that a uniformly chosen permutation π has breadth d is

$$e^{-\lambda} + O((\log n)^6 e^{\lambda}/n).$$

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The **minimum jump** (**mj**) of a permutation is the biggest distance between two adjacent entries of a permutation. It turns out this is easier to calculate, and translates well to breadth.

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Lemma

Fix a positive integer t. There exist functions $\{p_i(d)\}_{i=1}^{t}$ which are all at most polynomial, such that if $d = O(\log n)$, then

$$\mathbb{P}[\mathbf{mj} > d] = \left(1 + \sum_{i=1}^{t-1} \frac{p_i(d)}{n^i}\right) e^{-2d} + O\left(\frac{p_t(d)}{n^t} e^{2d}\right)$$

(Very Rough) Sketch

Let $\mathbf{br}(\pi)$ be the breadth of the random permutation π.

$$
\begin{aligned}
\mathbb{E}\left[\mathbf{br}(\pi)\right] &= -1 + \sum_{d=0}^{\lceil\sqrt{2n}\rceil} \mathbb{P}\left[\mathbf{br}(\pi) \geq d\right] \\
&= -1 + \sum_{d=0}^{\lceil(\log n)/2\rceil} \mathbb{P}\left[\mathbf{br}(\pi) \geq d\right] \\
&\quad + \sum_{\lceil(\log n)/2\rceil}^{\lceil\log n\rceil} \mathbb{P}\left[\mathbf{br}(\pi) \geq d\right] \\
&\quad + \sum_{\lceil\log n\rceil}^{\lceil\sqrt{2n}\rceil} \mathbb{P}\left[\mathbf{br}(\pi) \geq d\right]. \\
&= 2.13782018\ldots
\end{aligned}
$$

Conclusion

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Theorem

The distribution of close pairs is Poisson-λ with $\lambda = d^2 + d$, and so a random n-permutation has breadth $\geq d$ with probability approaching

$$e^{-d^2-d}.$$

Theorem

The expected breadth of a random permutation approaches

$$1 + \sum_{d \geq 0} e^{-d^2-d} \approx 2.13782018\ldots.$$

The expected minimum jump of a permutation is

$$\sum_{d \geq 0} e^{-2d} \approx 1.156517\ldots.$$

Complexity

Further, these ideas lead to an algorithm for measuring breadth which is $O\left(n^{3/2}\right)$ in the worst case, and $O(n)$ on average.

Thanks!