Stack sorting with increasing and decreasing stacks

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Permutation Patterns 2018, Dartmouth College, 9-13 July 2018.

Stack sorting (and relatives...)

General formulation:

INPUT - a permutation π;

MACHINE - a network of devices (may be stacks, queues, etc...), connected in series or in parallel (or in some more fancy way...);

OUTPUT - another permutation $f(\pi)$, which is hopefully the identity, otherwise somehow "closer" to the identity than the original permutation π.

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- Characterize the permutations that can be sorted by a given network.
- Enumerate sortable permutations with respect to their length.
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The set of permutations which can be sorted by a given network is a permutation class.

This is no longer true if we impose restrictions on the procedure, i.e. if we choose a specific algorithm to be used for the given network (e.g., West-2-stack sortable permutations).

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Our starting point: the machine $D^k I$

$k+1$ stacks in series:

- the first k stacks are decreasing (i.e. elements are maintained in decreasing order from top to bottom);
- the last stack is increasing.

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Results on the *DI* machine

Theorem (Smith, 2014)

The permutations sorted by a decreasing stack followed by an increasing one form the class $Av(3241, 3142)$.

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Operations of the $D^k I$ machine

- d_0: push the next element of the input permutation into the first decreasing stack D_1;
- d_i, for $i = 1, \ldots, k-1$: pop an element from D_i and push it into the next decreasing stack D_{i+1};
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Legal operation: when it does not violate the restrictions on the stacks.

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k-sortable permutations

$$B(k) = \{\pi \in S \mid \text{ there is a sequence of legal operations } d_{i_1}, \ldots, d_{i_s} \text{ that sorts } \pi\}$$

Goal: understand the basis of $B(k)$.

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The case $k = 2$

Theorem

For $j \geq 0$, define the permutation:

$$\alpha_j = 2j+4, 3, a_1, b_1, a_2, b_2, \ldots, a_j, b_j, 1, 5, 2$$

where:

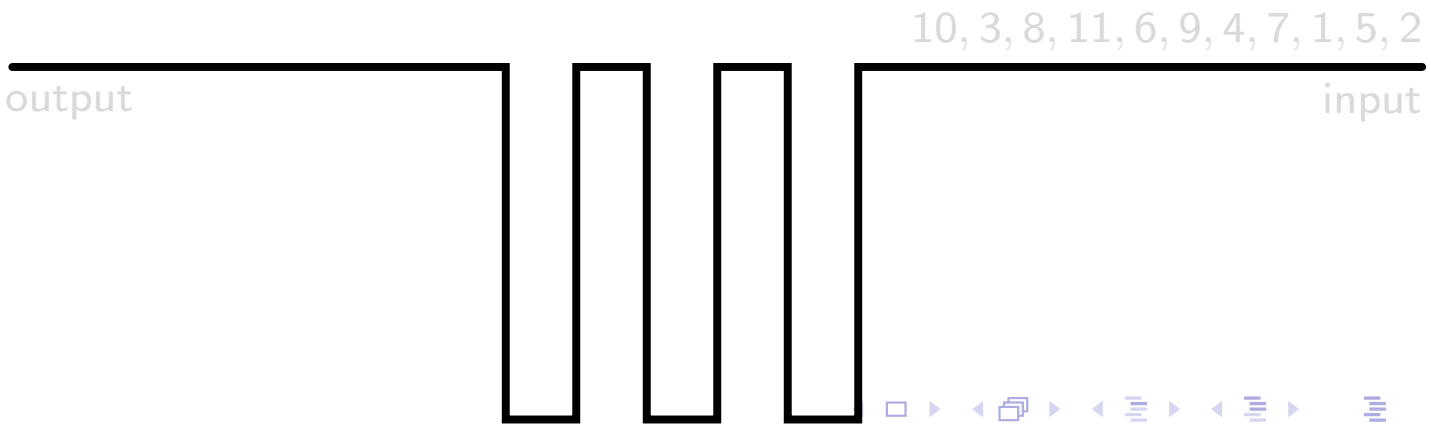
$$\begin{cases} A_j = (a_1, \ldots, a_j) = (2j+2, 2j, 2j-2, \ldots, 6, 4), \\ B_j = (b_1, \ldots, b_j) = (2j+5, 2j+3, 2j+1, \ldots, 9, 7). \end{cases}$$

Then the set of permutations $\{\alpha_j\}_{j\geq 0}$ constitutes an infinite antichain each of whose elements is not 2-sortable. Moreover, α_j is minimal with respect to such a property, i.e. if we remove any element of α_j we obtain a 2-sortable permutation. As a consequence, the basis of $B(2)$ is infinite, since it contains the infinite antichain $\{\alpha_j\}_{j\geq 0}$.

Proof

α_j is not 2-sortable: induction on j.

- $j = 0$: the permutation 43152 is not 2-sortable.
- Generic j:



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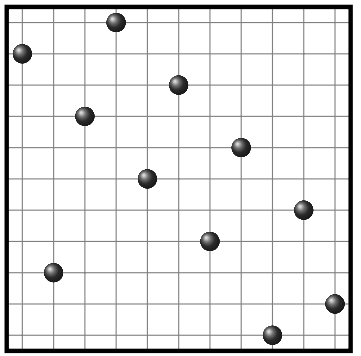

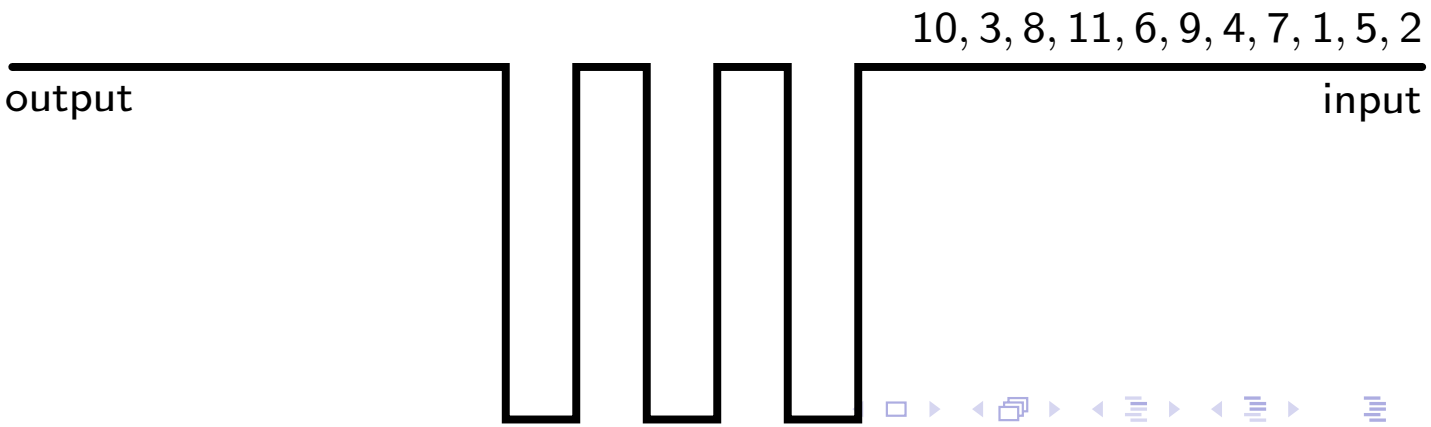

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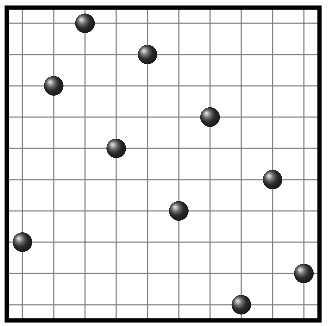

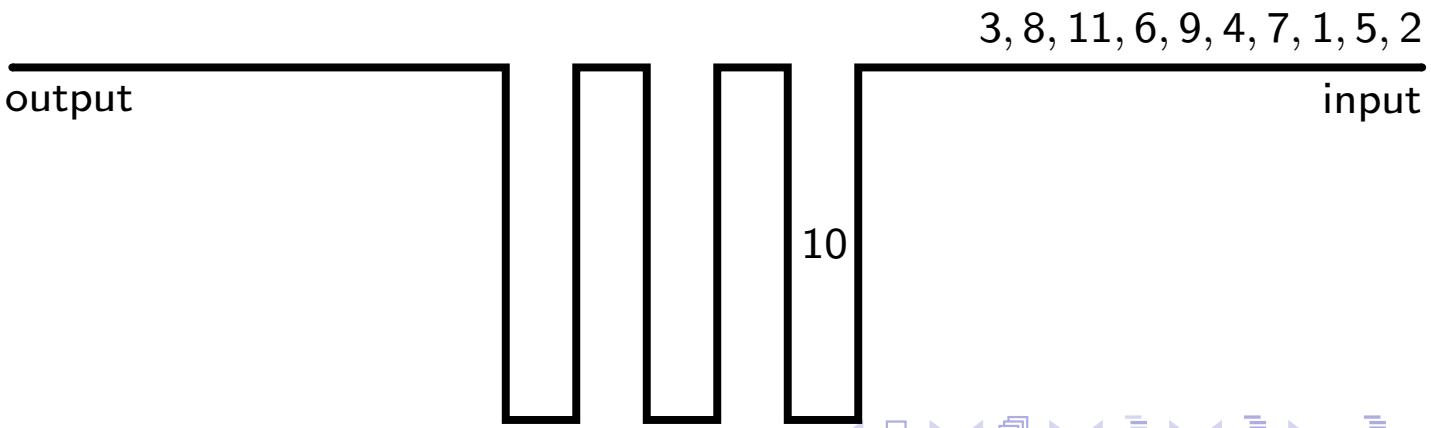

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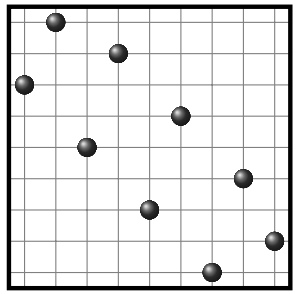

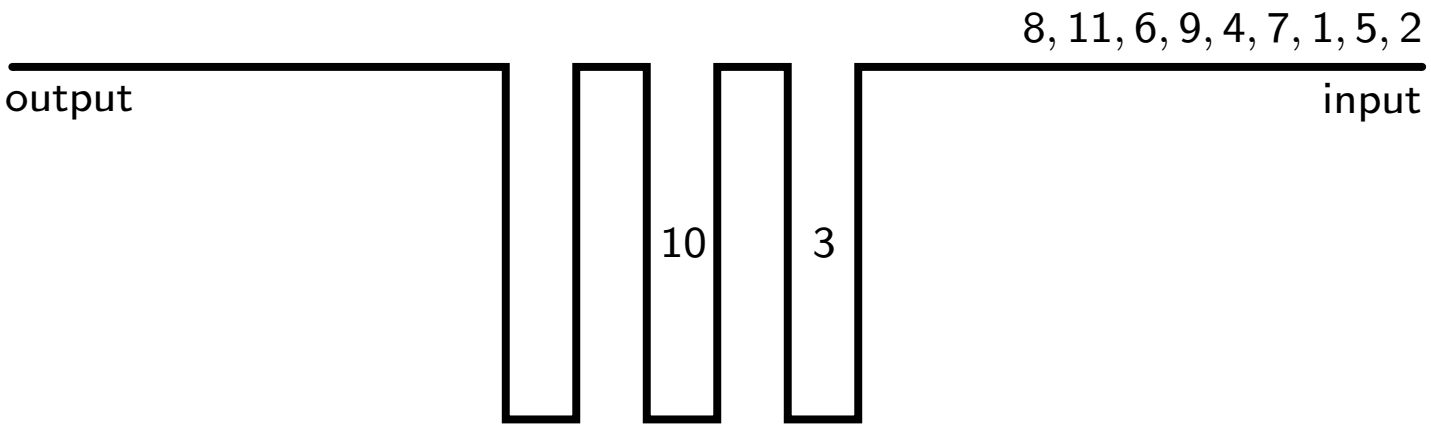

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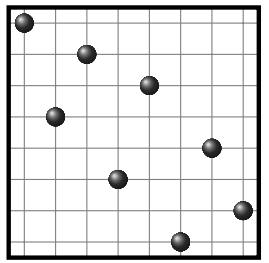

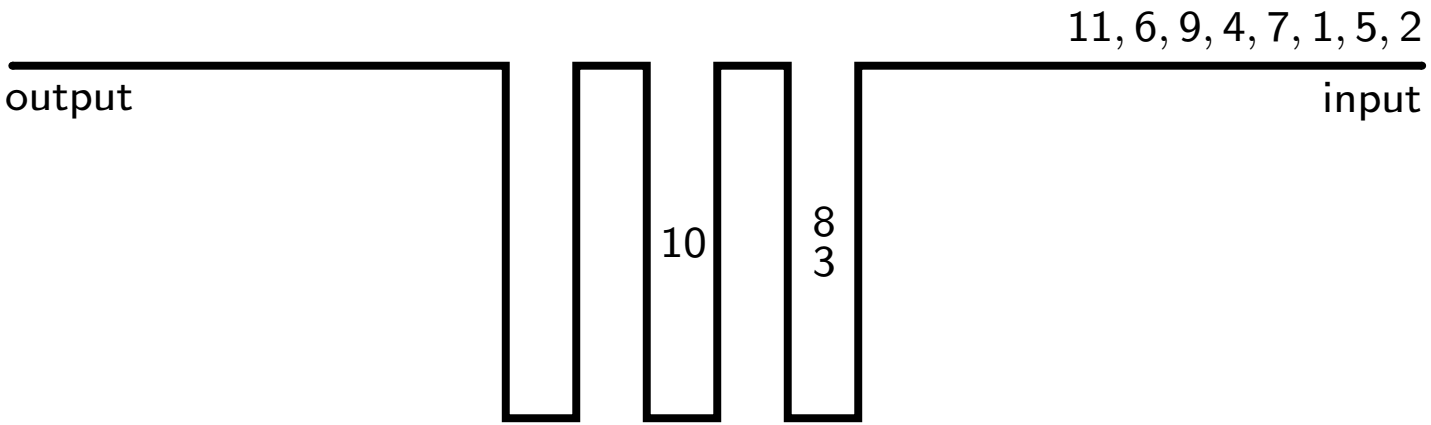

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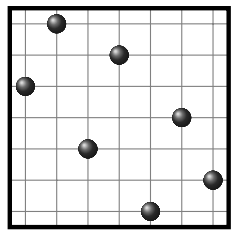

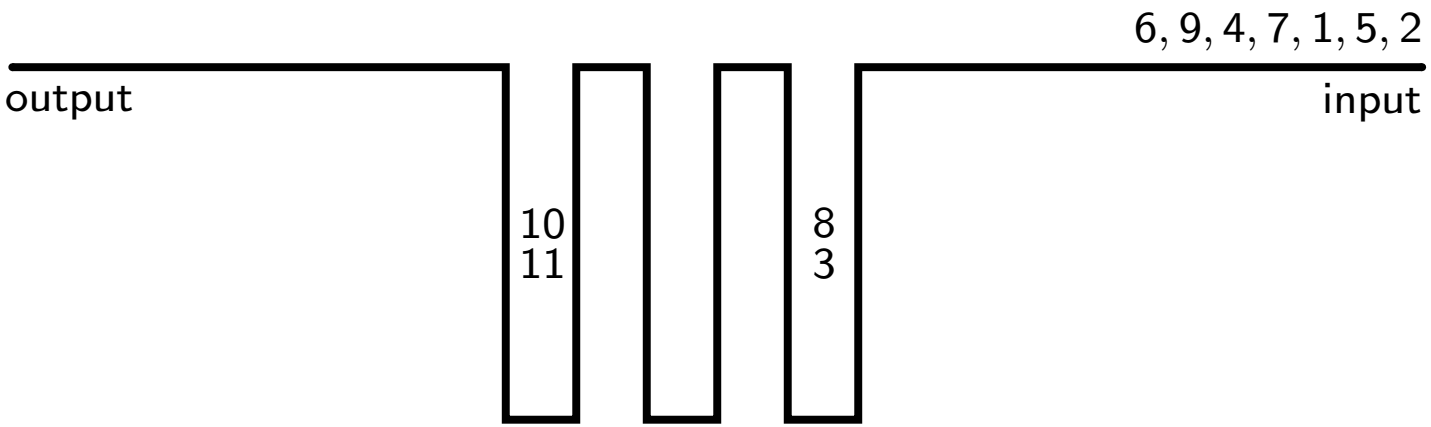

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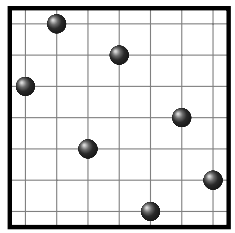

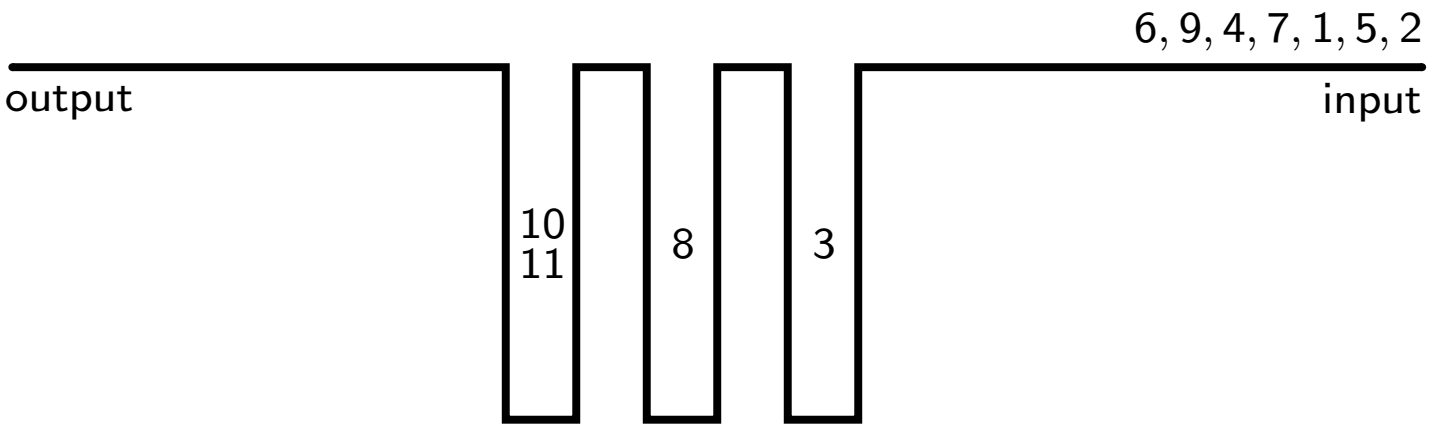

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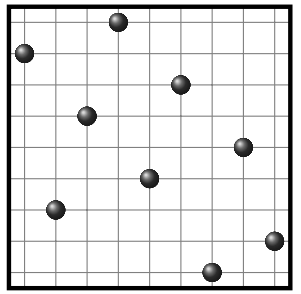

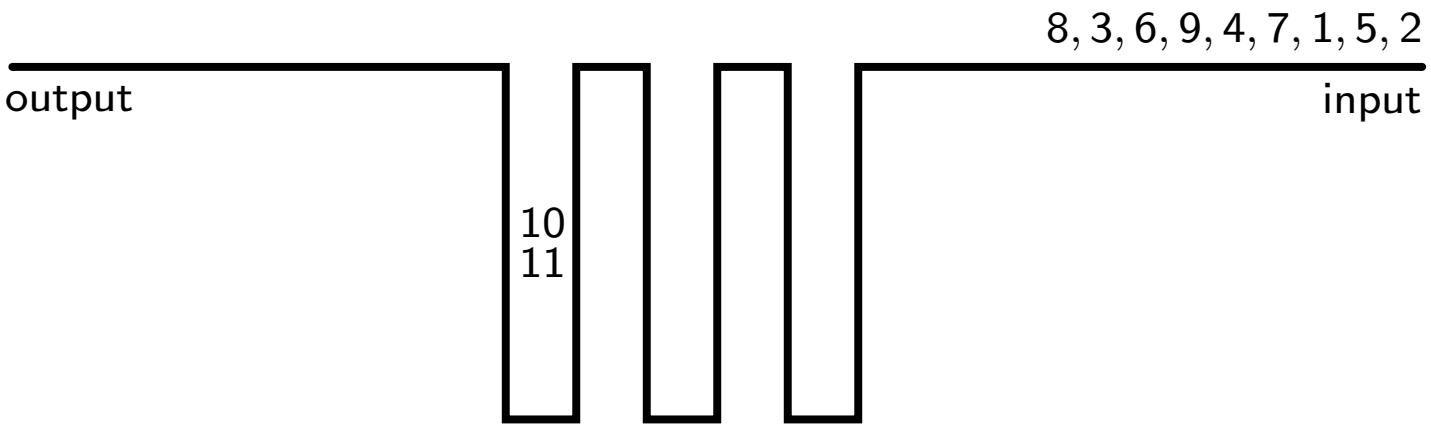

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- Remove $2j + 4$.
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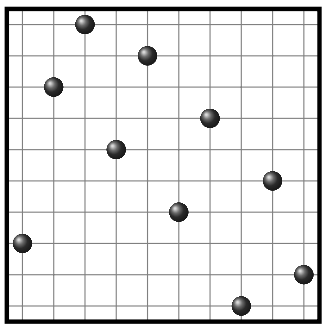

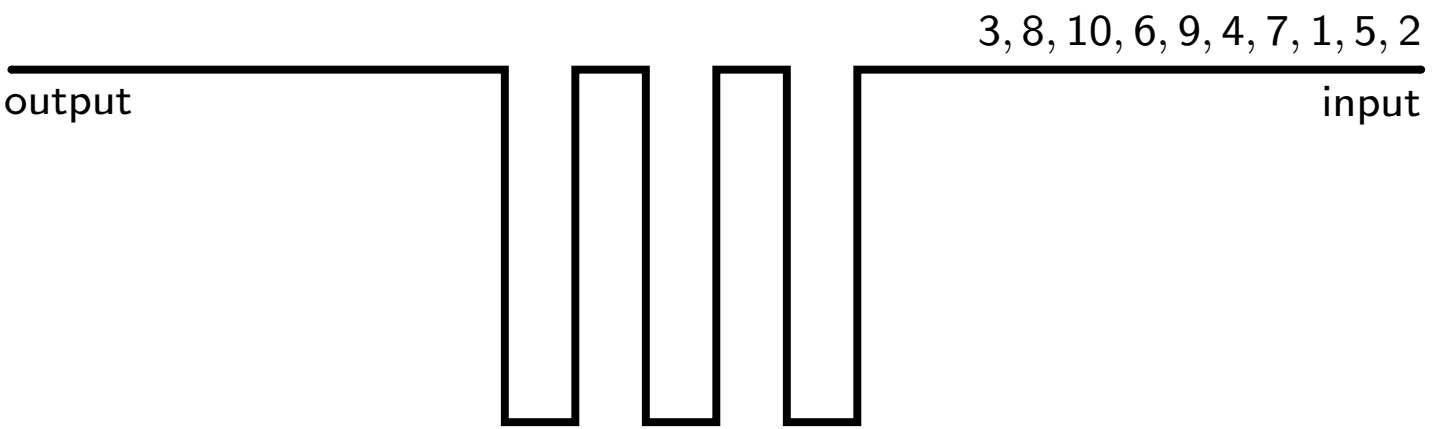

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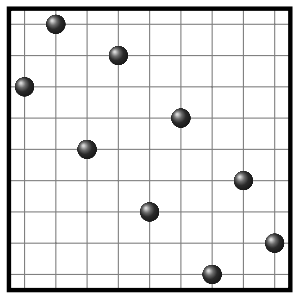

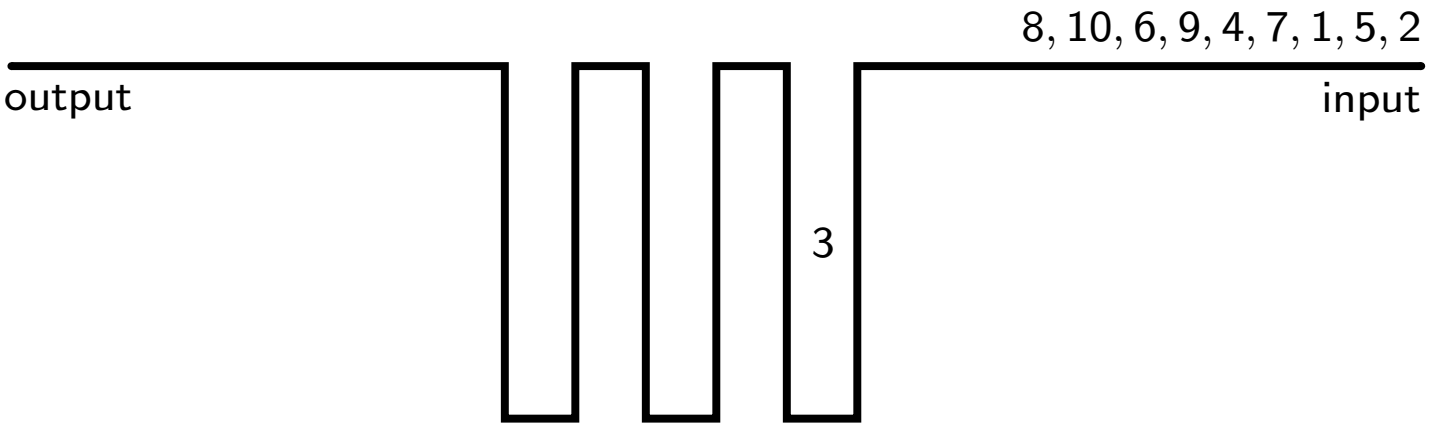

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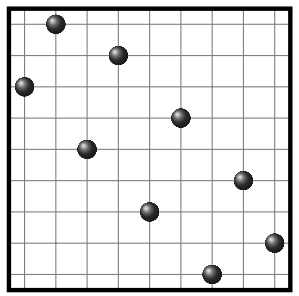

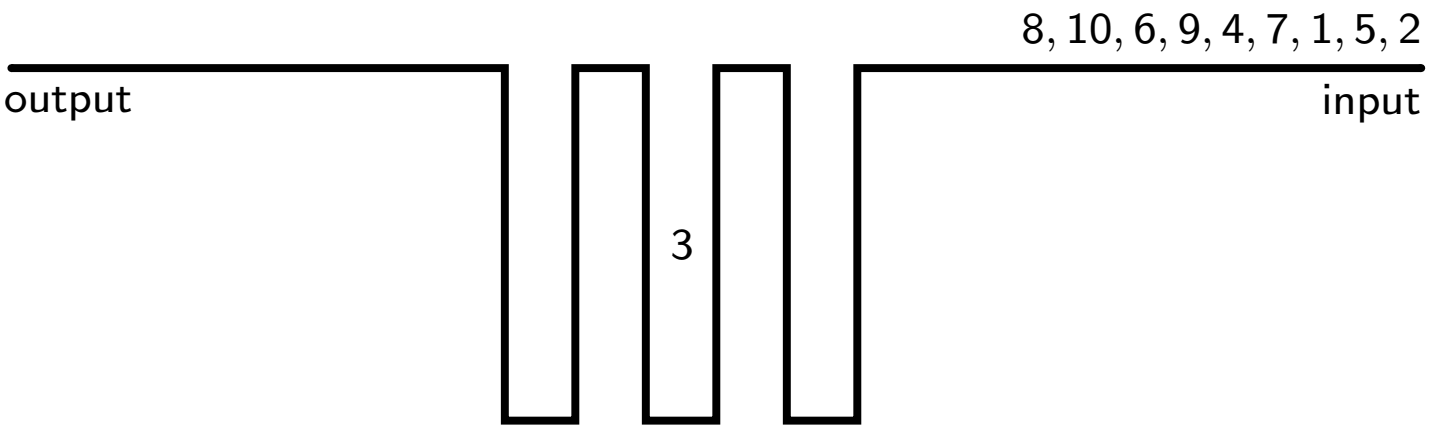

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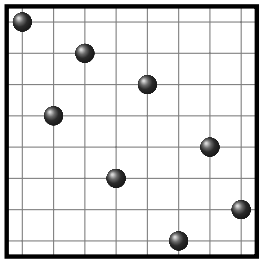

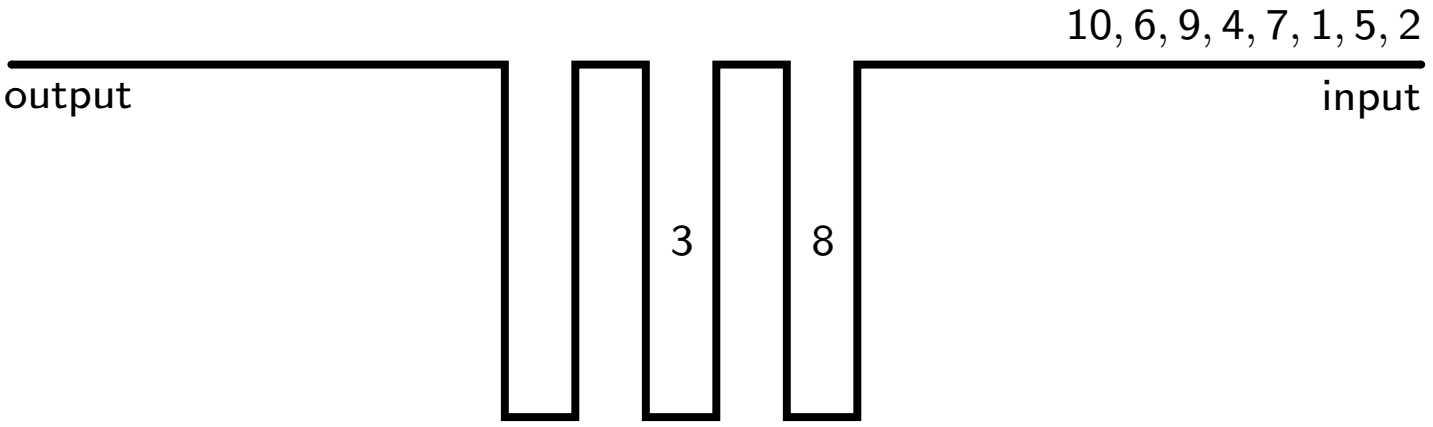

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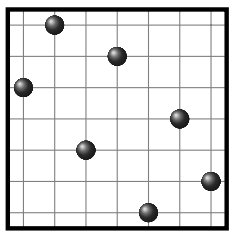

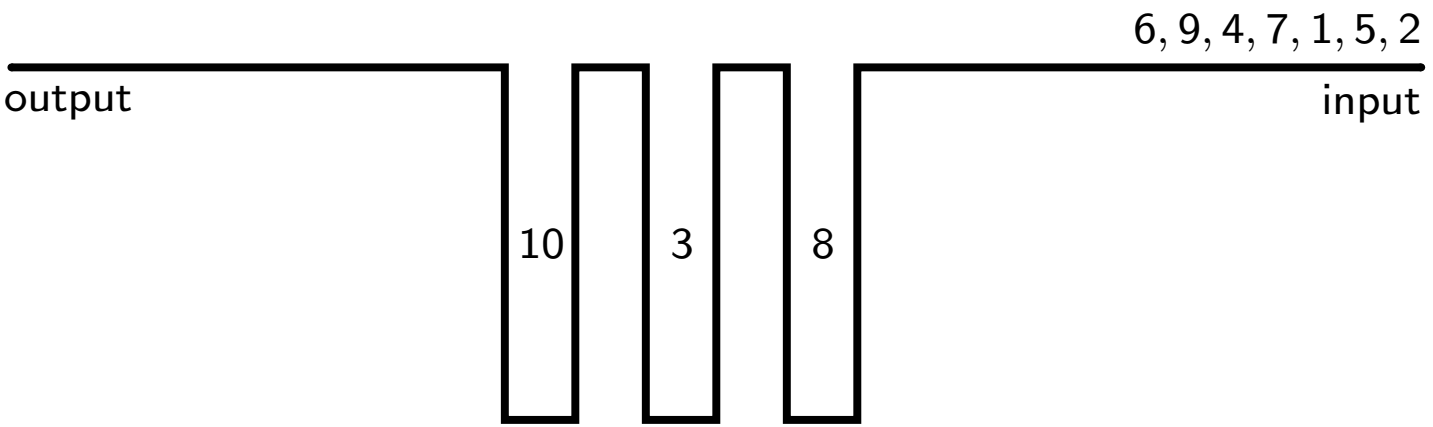

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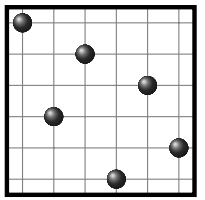

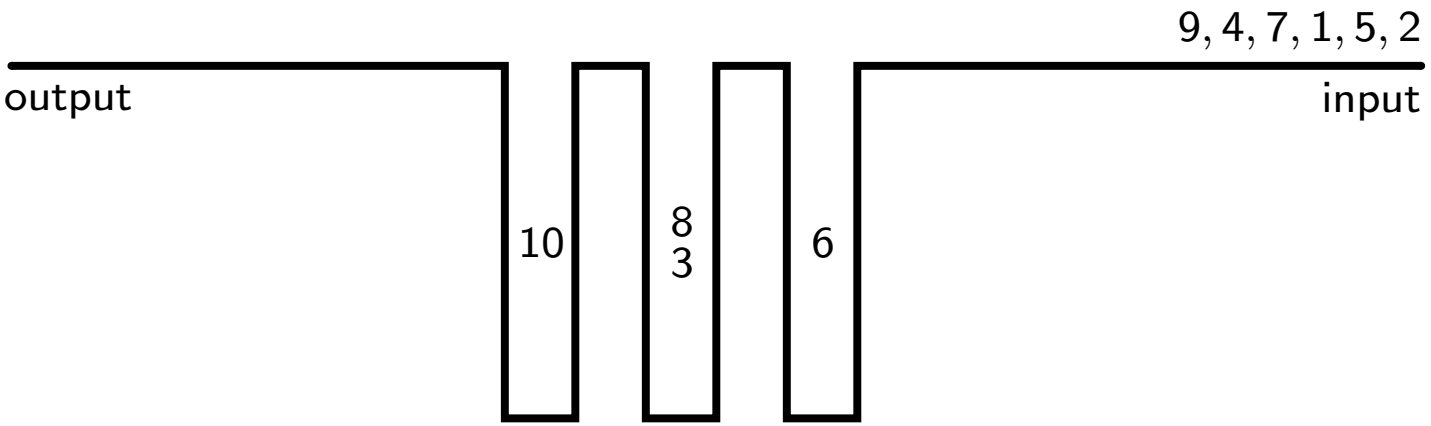

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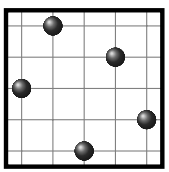

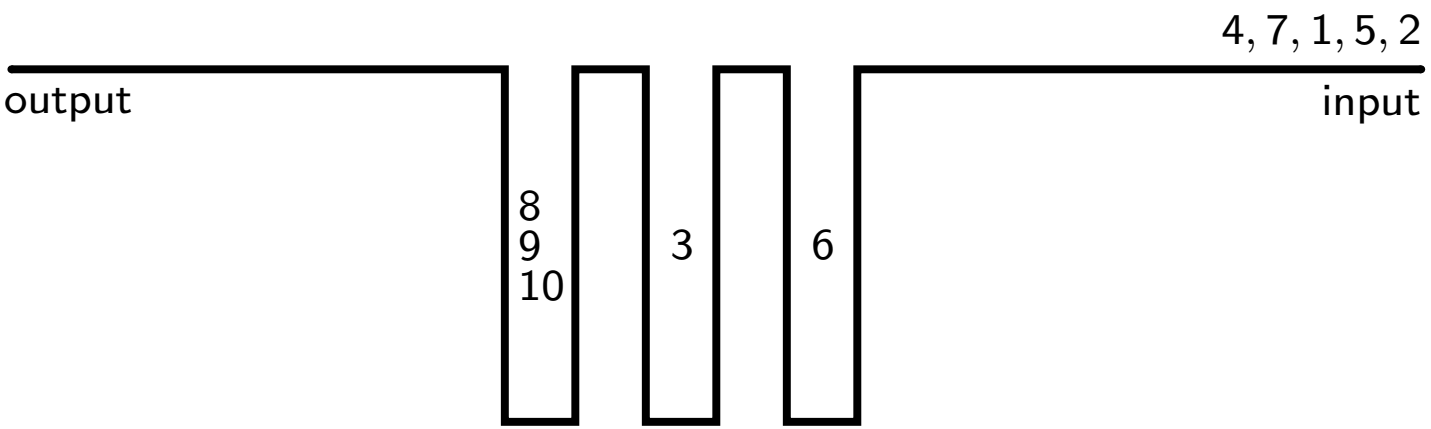

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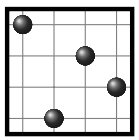

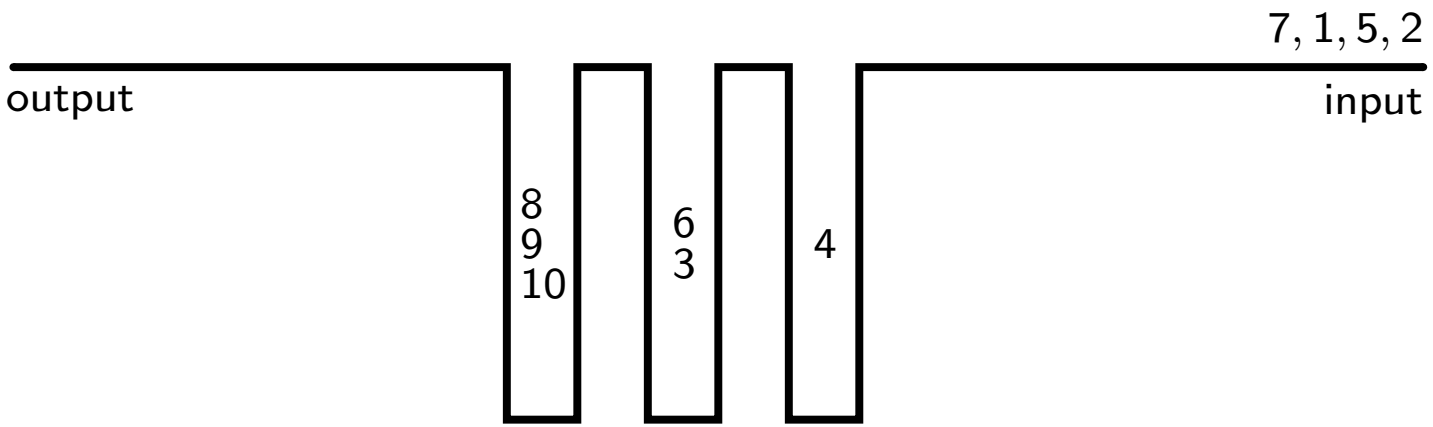

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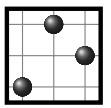
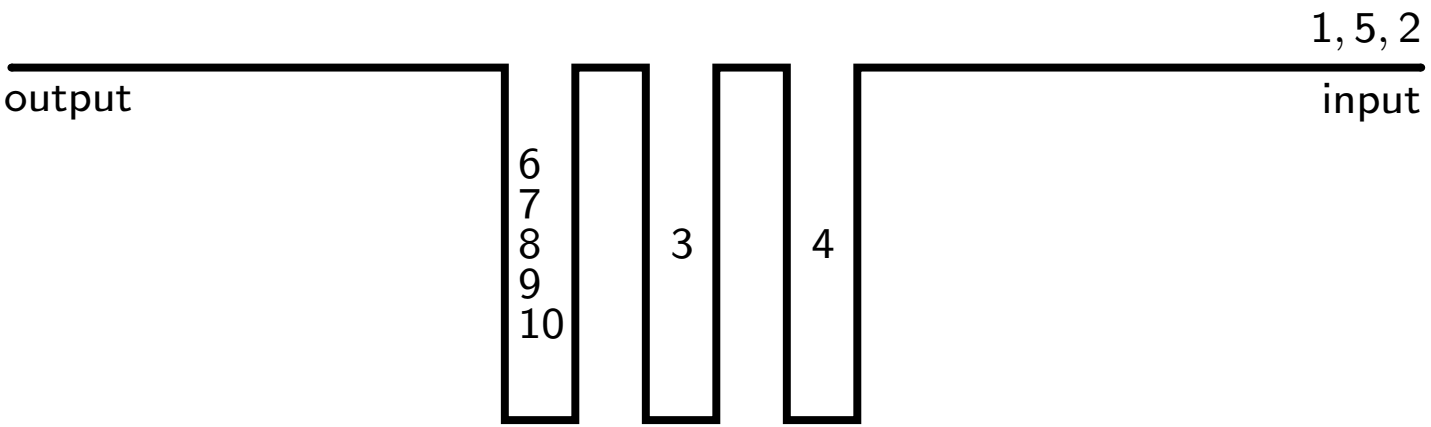
All patterns of α_j are 2-sortable: case-by-case analysis.
Remove $2j + 4$:





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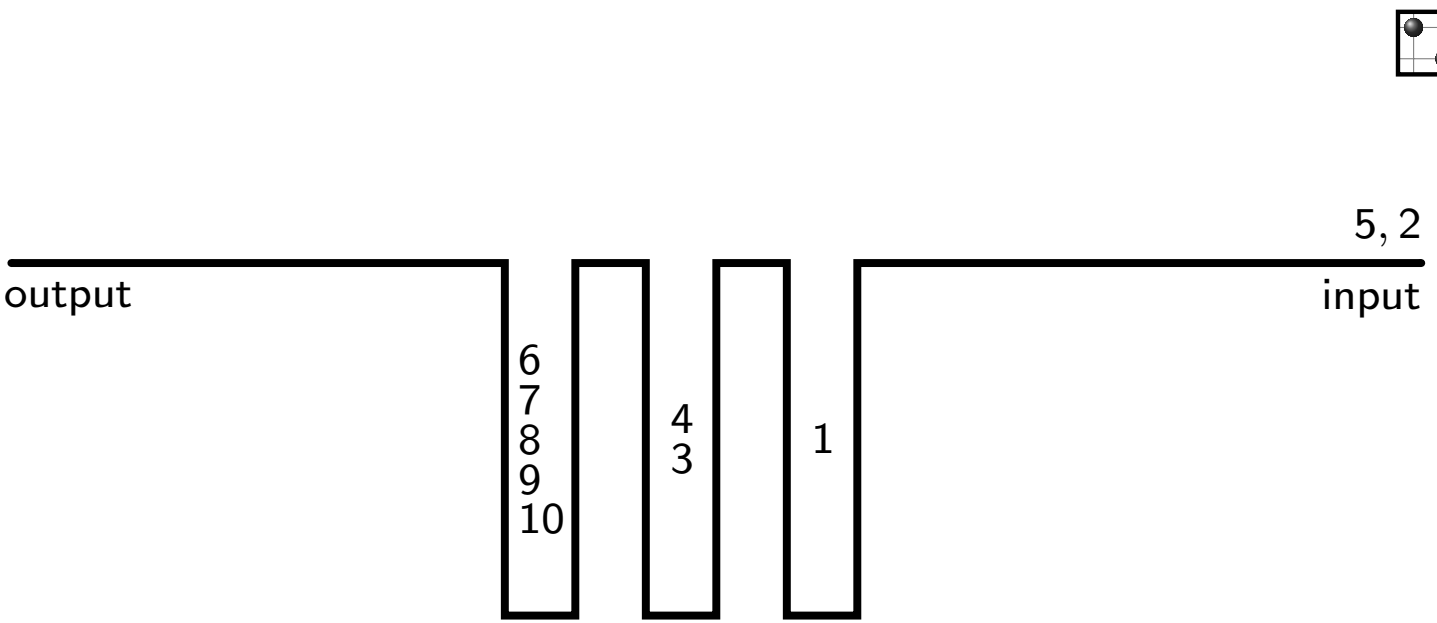

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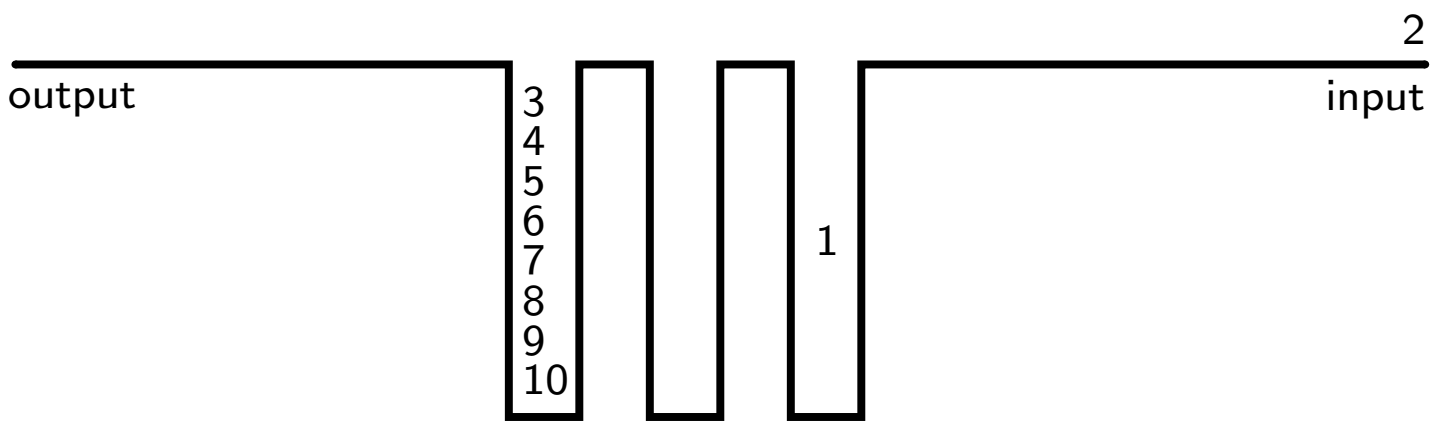

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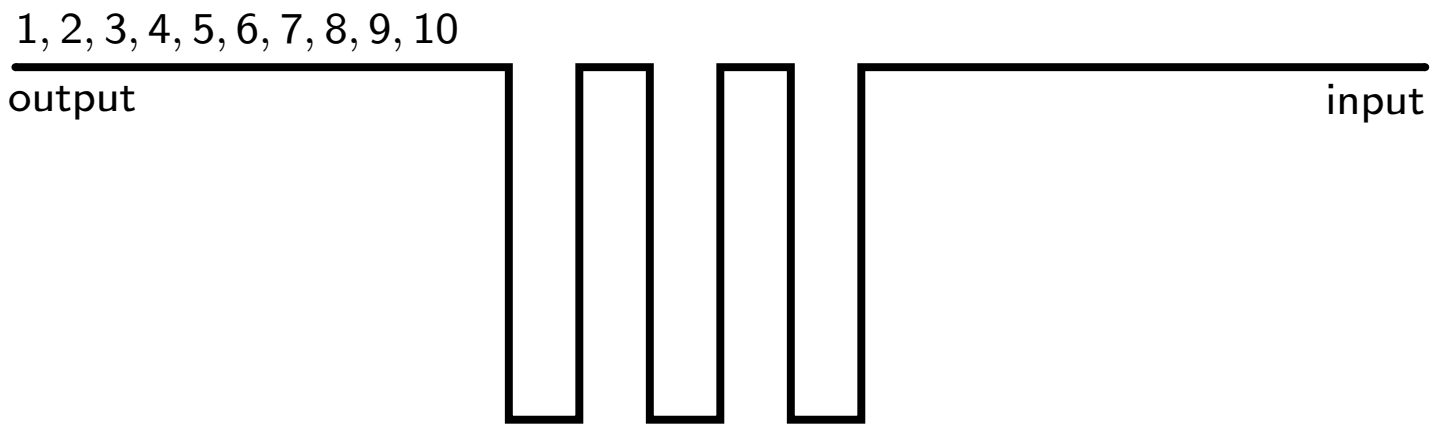

First algorithm: left greedy

Priorities of operations: $d_i \rhd d_j$ whenever $i > j$.

$B_{lg}(k) = \{\pi : \pi \text{ is sorted by the } D^k I \text{ machine using the left-greedy procedure}\}$

Proposition

For every $k \geq 1$, $B_{lg}(k) = Av(231)$.

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Comparison with Stacksort

Thus the left greedy algorithm sorts precisely the same permutations as Stacksort does.

However, the two algorithms are not equivalent: for instance, when $k = 1$, on the input permutation 2341:

- the left greedy DI machine outputs 2134;
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Comparison with Stacksort

$$\phi_k : S_n \longrightarrow S_n$$

$\phi_k(\pi) =$ the permutation which exits the last (i.e., the k-th) decreasing stack.

Of course, ϕ_k preserves the property of being a 231-avoider.
What further properties does ϕ_k have?
For instance, for any given π, the sequence $\{\phi_k(\pi)\}_{k\in\mathbb{N}}$ eventually becomes constant. But we do not know when this happens precisely.
For $\pi = 36257418$:

$$k = 1 : 36275418,$$
$$k = 2 : 37652841,$$
$$k = 3 : 37652841,$$
$$k = 4 : 38765241,$$
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Second algorithm: almost left greedy

Priorities of operations: $d_{k+1} > d_{k-1} > d_{k-2} > \cdots > d_1 > d_0 > d_k$.
This means that the algorithm always performs the leftmost possible operation, except for the push operation into the increasing stack, which is performed last.

$B_{alg}(k) = \{\pi : \pi \text{ is sorted by the } D^k I \text{ machine using the almost left-greedy procedure}\}$

Unfortunately, $B_{alg}(k)$ is not in general a class: for $k = 2$, 631425 is sortable, but its pattern 52314 isn't.

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The case $k = 2$

For the almost left greedy D^2I machine we only have some partial results.

Proposition

Let π be an almost left-greedy D^2I sortable permutation; then:

- *π avoids 3214;*
- *π avoids the following barred patterns, each of which is obtained by suitably adding barred elements to the pattern 52314:*
  - $63\bar{1}425$;
  - $7\bar{2}\bar{1}4536$, $7\bar{3}\bar{1}4526$;
  - $\bar{7}\bar{2}8\bar{1}4536$, $\bar{7}\bar{3}8\bar{1}4526$;
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Let π be a permutation that is not almost left-greedy D^2I sortable. Then one of the following cases holds:

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  - 1021493675 8 (resp., 1021593674 8);
  - 1031492675 8 (resp., 1031592674 8);
  - 10211493675 8 (resp., 10211593674 8);
  - 10311492675 8 (resp., 10311592674 8);
  - 11210149367 58 (resp., 10211593674 8);
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In fact, we can generate an infinite sequence of permutations $(\gamma_n)_{n\in\mathbb{N}}$, with $\gamma_n \in S_{3n+2}$, such that $\gamma_n \leq \gamma_{n+1}$ for all n, and permutations having even index are sortable, whereas permutations having odd index are not.

Formally,

$$\gamma_n = 3n+2,\ \underbrace{2\ 3n+1\ 1}_{231},\ \underbrace{4\ 3n\ 3}_{231}\cdots\underbrace{2n-2\ 2n+3\ 2n-3}_{231}\ \underbrace{2n\ 2n+1\ 2n-1}_{231}\ 2n+2,$$

and we have:

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⋮

The case $k = 2$

In fact, we can generate an infinite sequence of permutations $(\gamma_n)_{n\in\mathbb{N}}$, with $\gamma_n \in S_{3n+2}$, such that $\gamma_n \leq \gamma_{n+1}$ for all n, and permutations having even index are sortable, whereas permutations having odd index are not.

Formally,

$$\gamma_n = 3n+2,\ \underbrace{2\ 3n+1\ 1}_{231},\ \underbrace{4\ 3n\ 3}_{231}\cdots\underbrace{2n-2\ 2n+3\ 2n-3}_{231}\ \underbrace{2n\ 2n+1\ 2n-1}_{231}\ 2n+2,$$

and we have:

- $\gamma_1 = 52314 \notin B_{alg}(2)$;
- $\gamma_2 = 82714536 \in B_{alg}(2)$;
- $\gamma_3 = 11\ 2\ 10\ 14936758 \notin B_{alg}(2)$;

$\vdots$

What next?

There are several interesting things that are still to explore.

- Smarter algorithms for the $D^k I$ machine? An optimal one (at least in the $k = 2$ case)?
- What about making two passes from Rebecca's DI machine? Analogies with West-2-stack-sortable permutations?
- Enumerations?

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