Knots and Permutations

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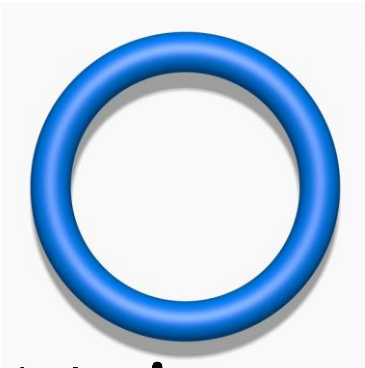
Nati Linial, Tahl Nowik

Knots



Unknot

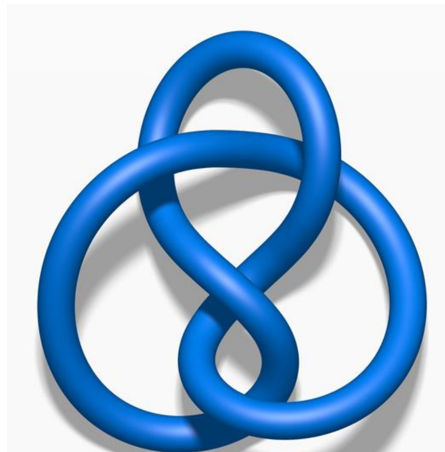
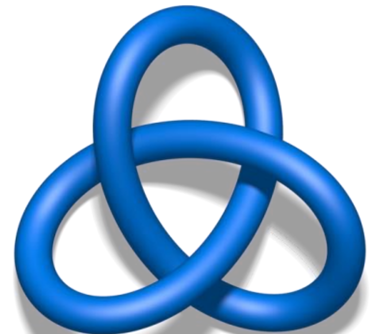
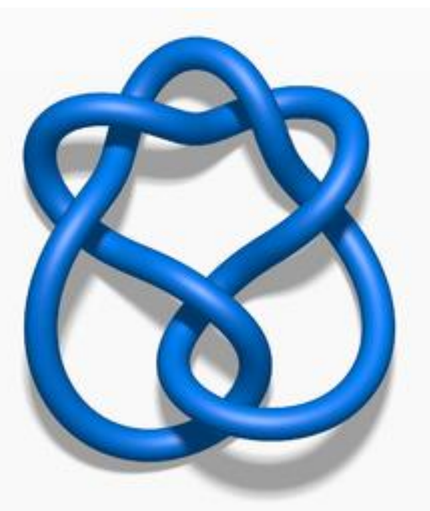
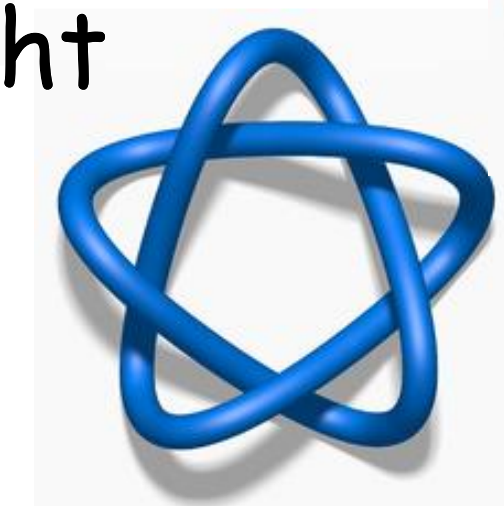


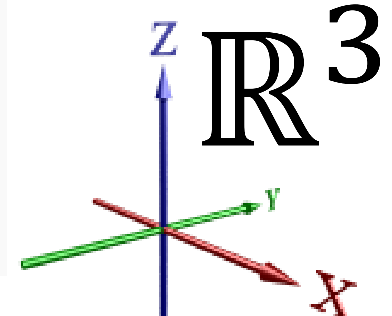
Figure-eight



Trefoil







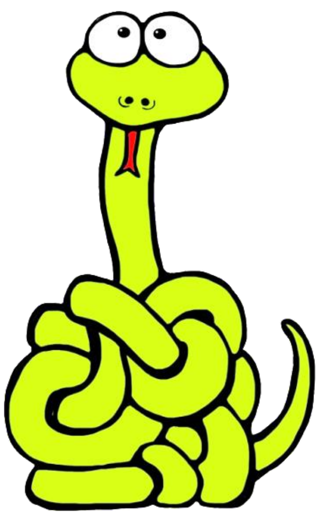
Knot Theory

How to represent knots?

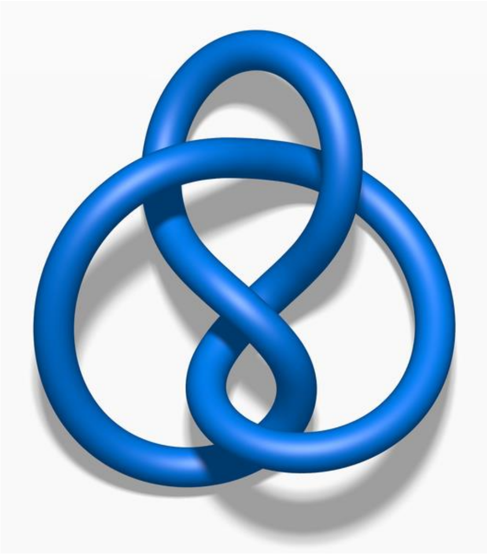
show that two knots are the same?

tell non-equivalent knots apart?

study properties of knots?

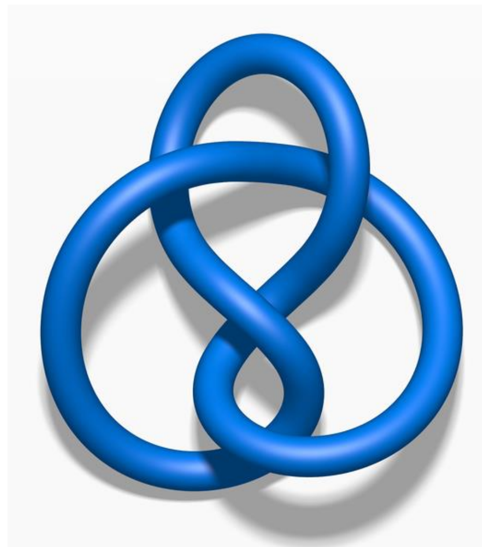


Knot Diagrams

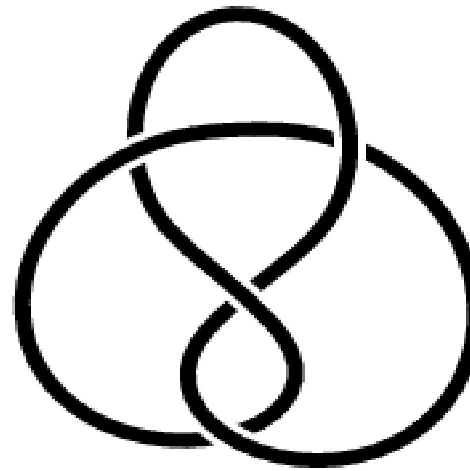


$$S^1 \xhookrightarrow[\text{[Knot]}]{} \mathbb{R}^3$$

Knot Diagrams

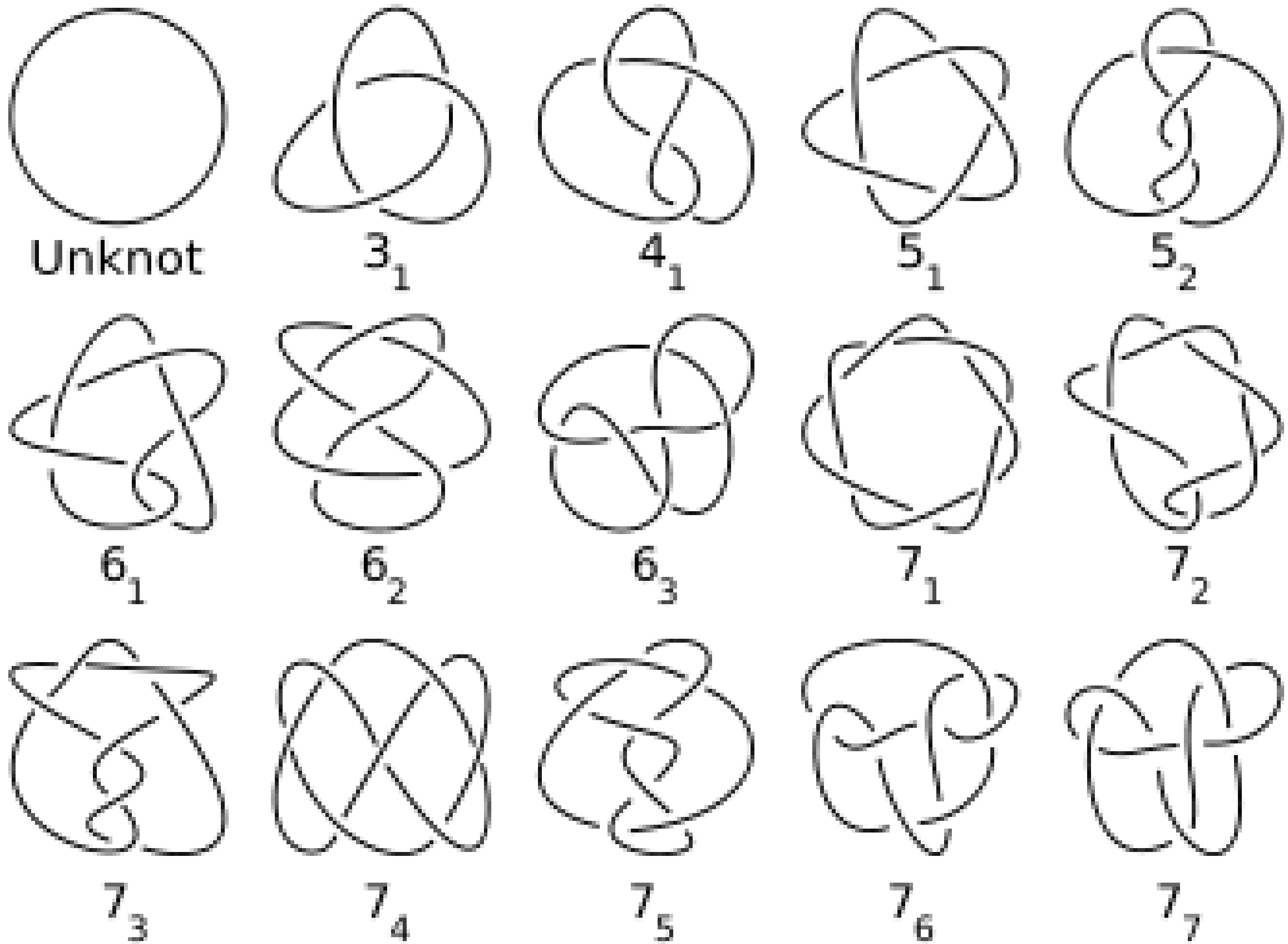


Knot Diagram

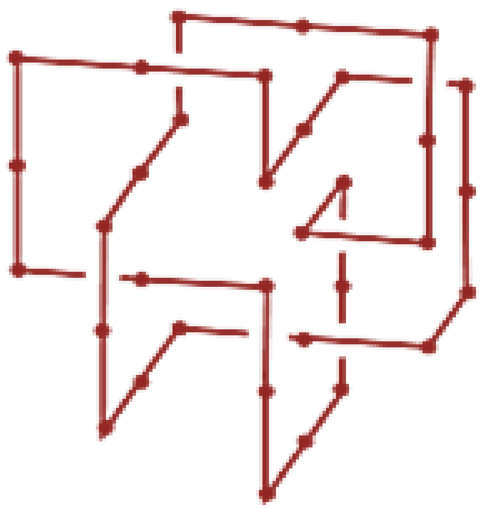


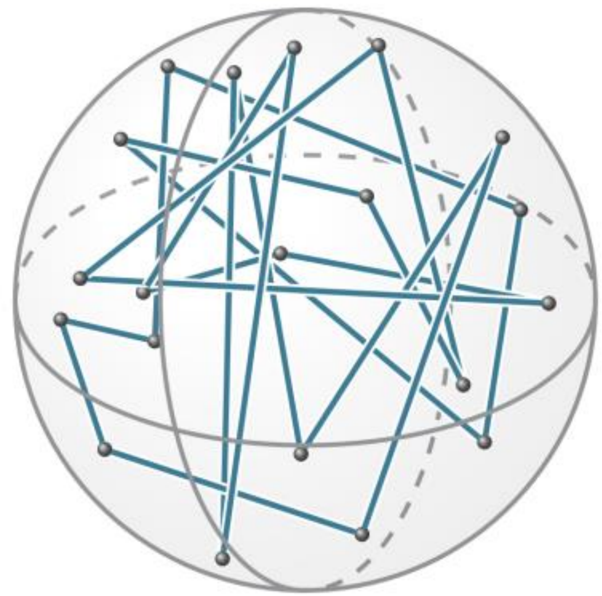
$$S^1 \underset{[\text{Knot}]}{\hookrightarrow} \mathbb{R}^3 \underset{\text{Projection}}{\longrightarrow} \mathbb{R}^2$$

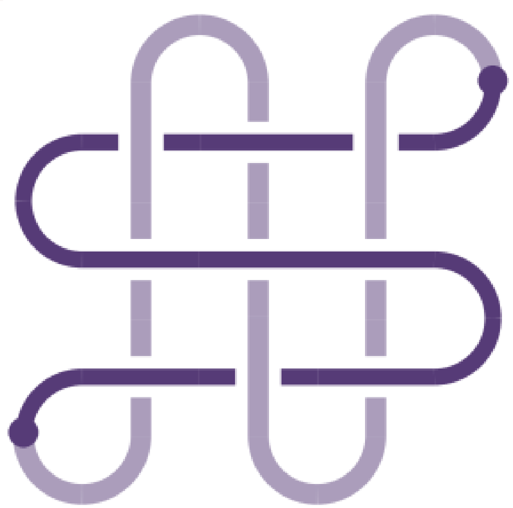
Knot Diagrams

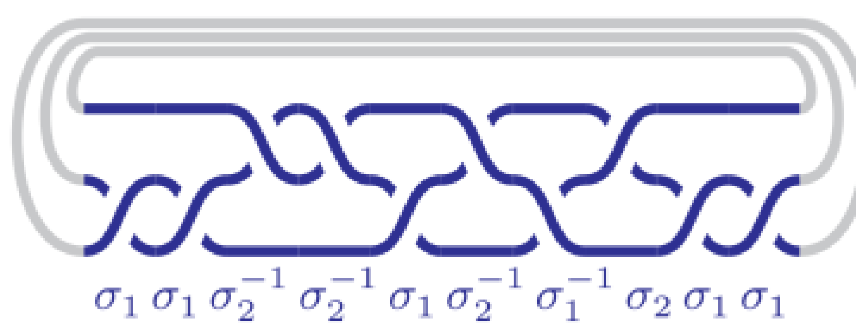


More Specific Representations



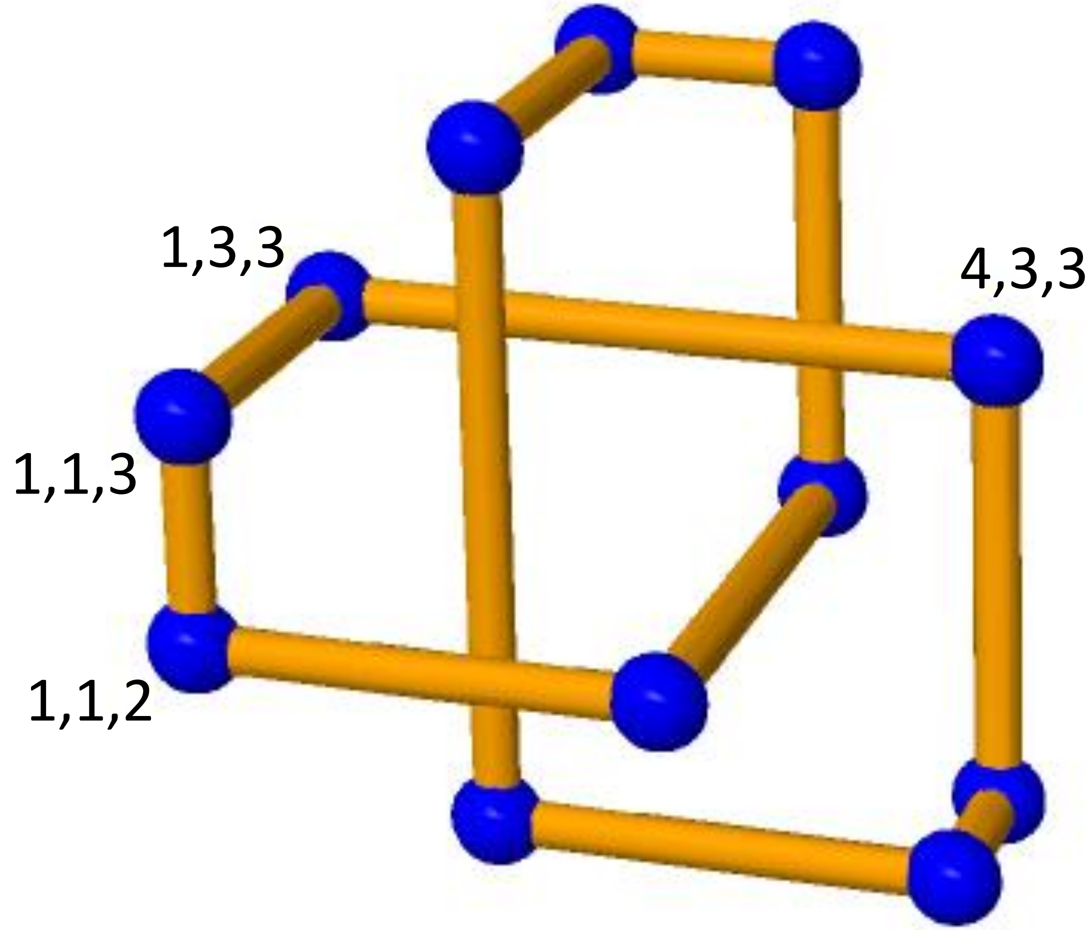






Knots from 3 Permutations

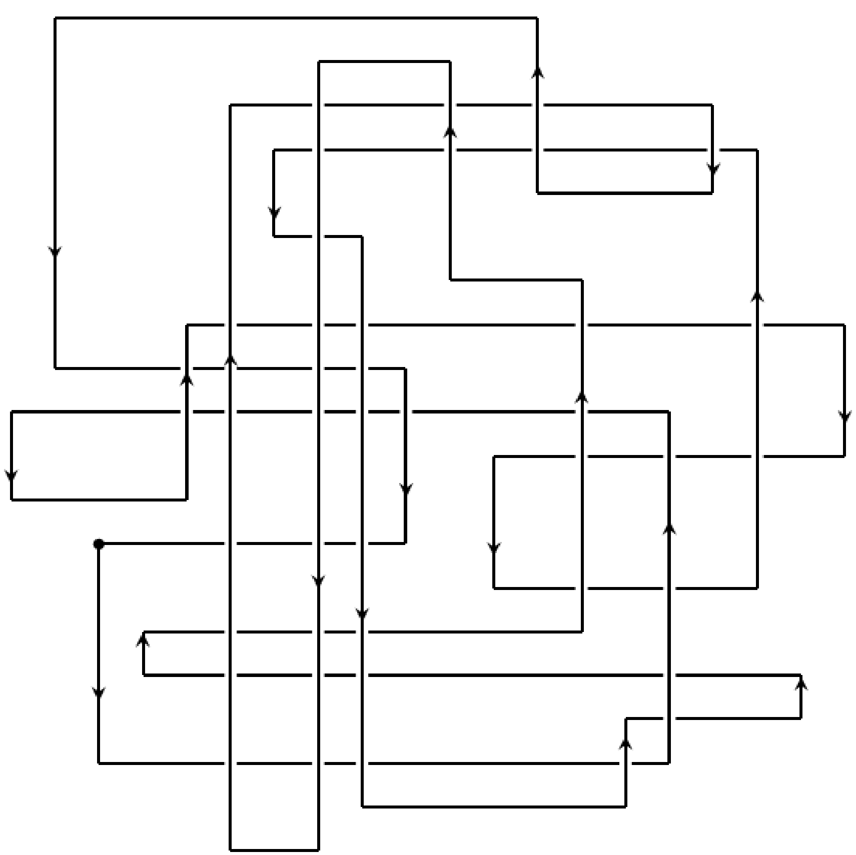
X = 1 4 2 3
Y = 1 3 2 4
Z = 2 3 1 4

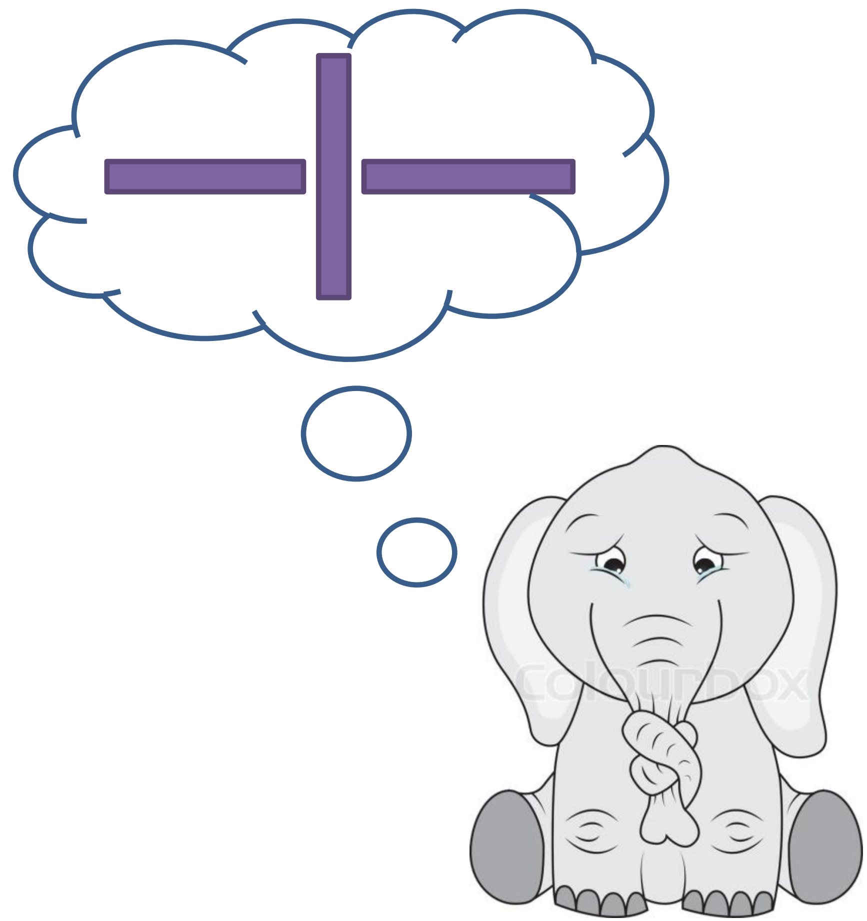


$(X_1,Y_1,Z_1) \rightarrow (X_1,Y_1,Z_2) \rightarrow (X_1,Y_2,Z_2) \rightarrow (X_2,Y_2,Z_2)$
$\rightarrow (X_2,Y_2,Z_3) \rightarrow (X_2,Y_3,Z_3) \rightarrow \ldots$

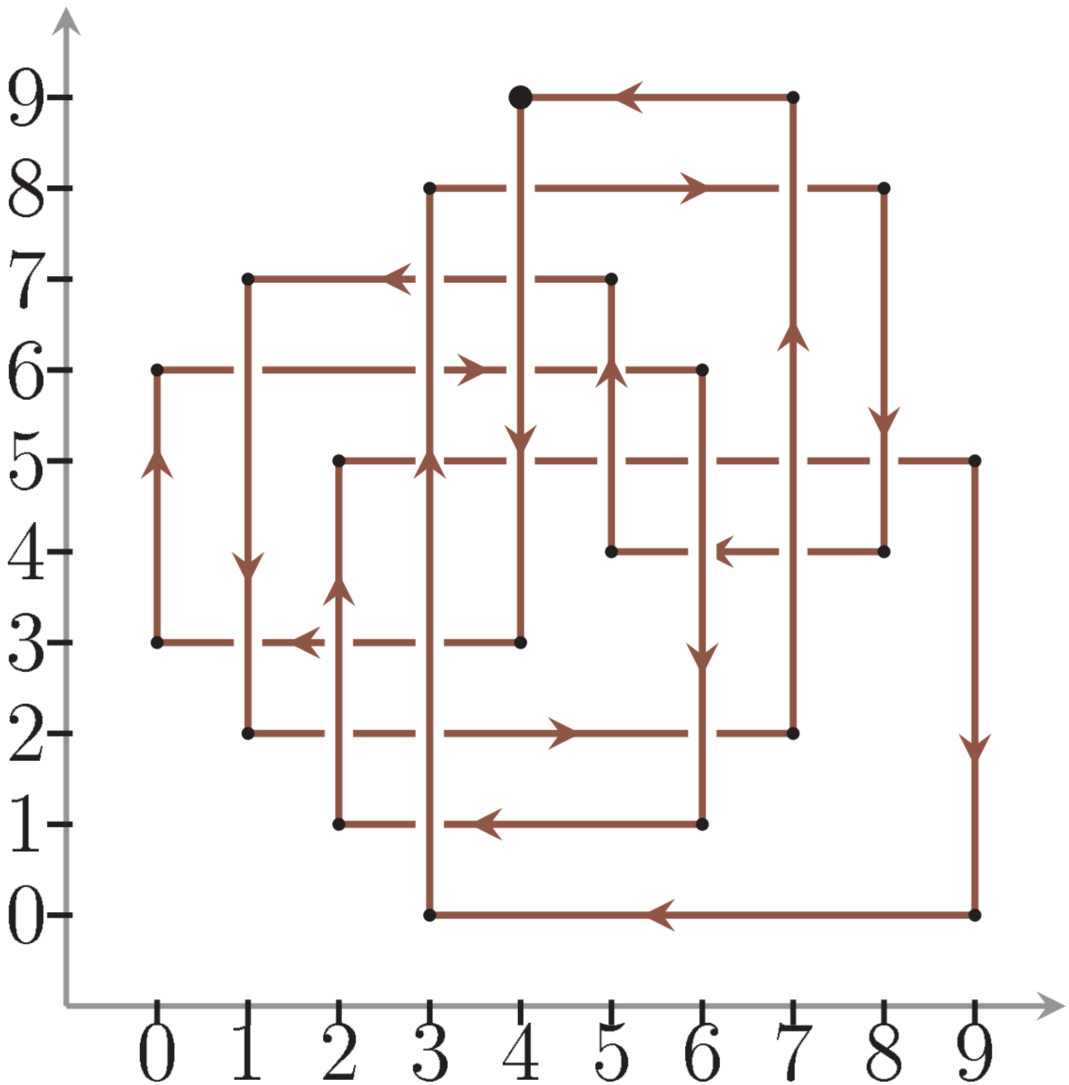
Grid Diagrams

Every **knot diagram** is isotopic to a **grid diagram**





Knots from Two Permutations

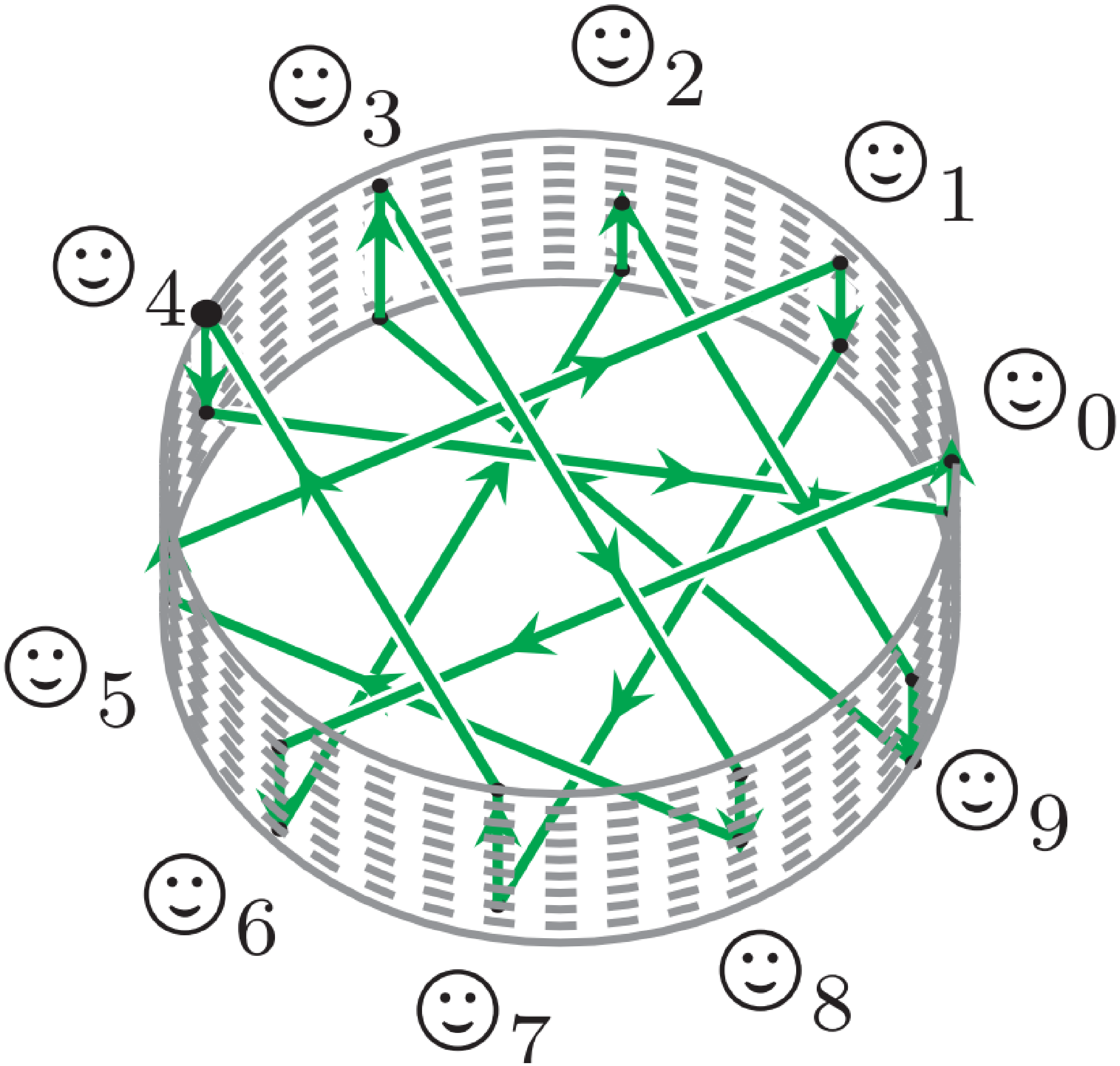


4 0 6 2 9 3 8 5 1 7

9 3 6 1 5 0 8 4 7 2

(4,9) -> (4,3) -> (0,3) -> (0,6) -> (6,6) -> ...

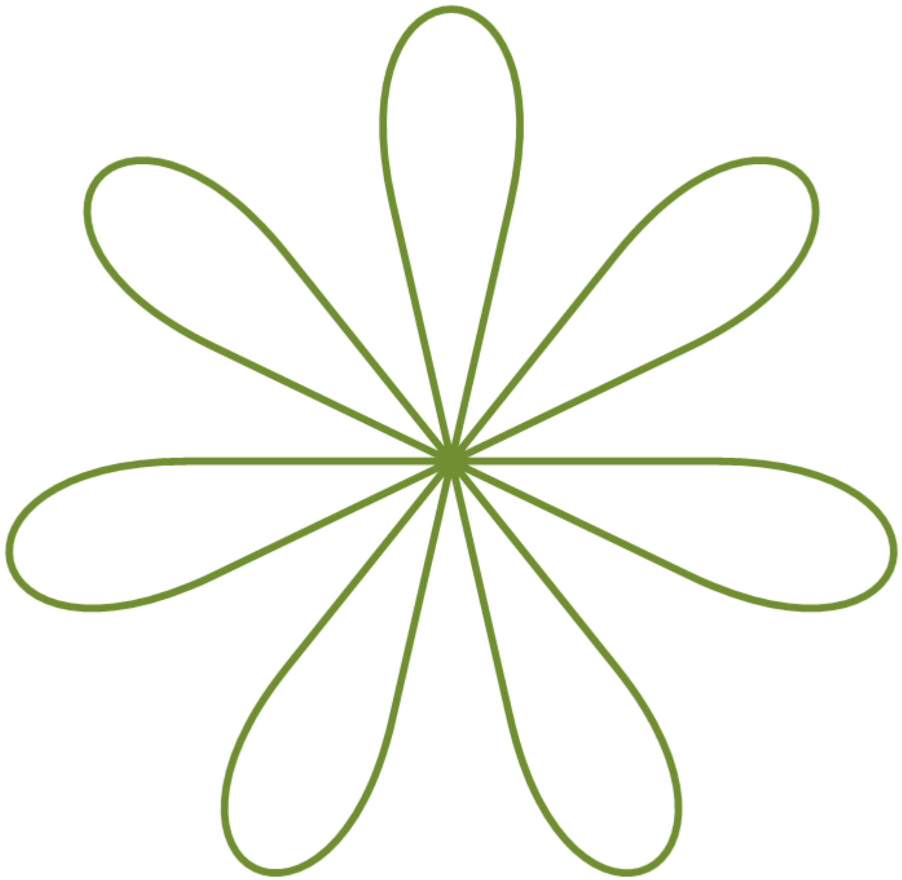
The Human Knot



4 0 6 2 9 3 8 5 1 7

9 3 6 1 5 0 8 4 7 2

Petal Diagrams

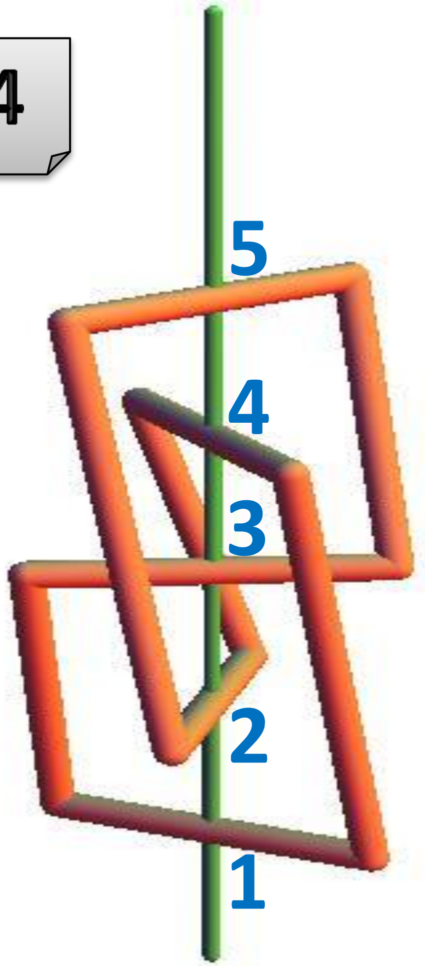


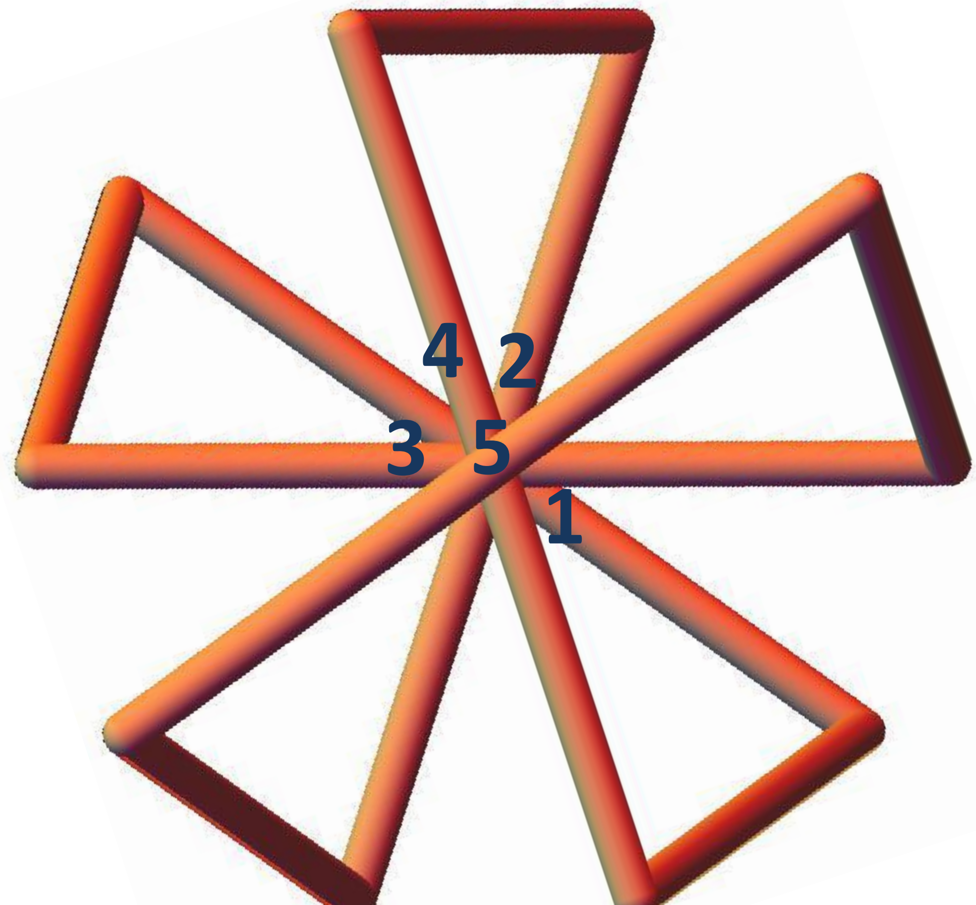
[Knot projections with a single multi-crossing. Adams, Crawford, De Meo, Landry, Lin, Montee, Park, Venkatesh, Yhee. 2012]

Projections

Knots from One Permutation

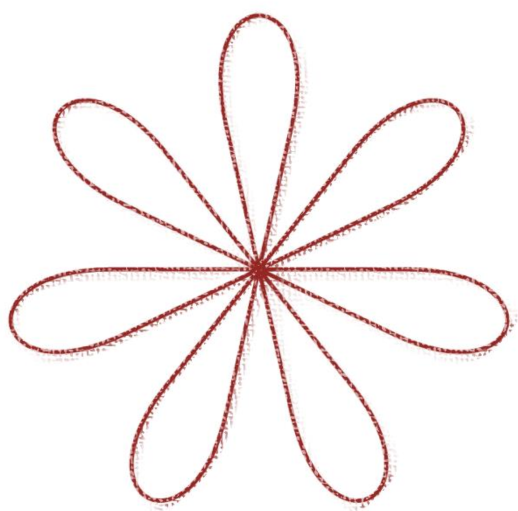
1 3 5 2 4

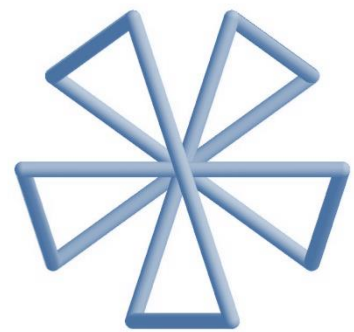


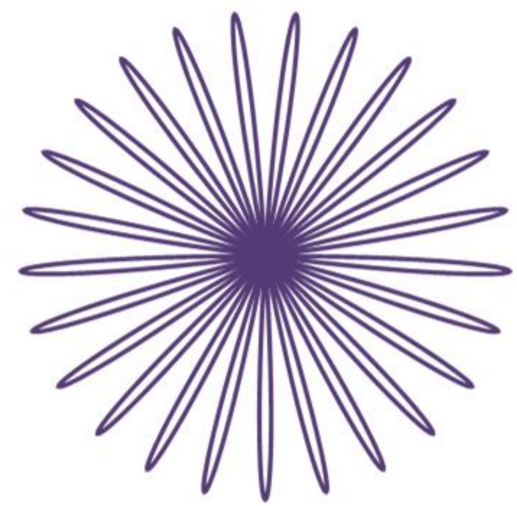


Theorem [Adams et al. 2015]

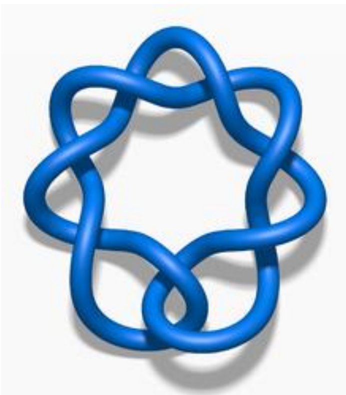
Every knot has a petal diagram.







Crossings V Petals



Twist knots

n crossings

n+2 or n+3 petals

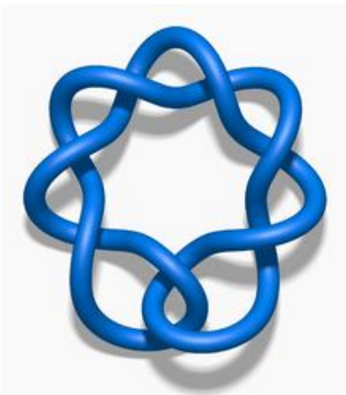
Torus(n,n+1)

n^2-1 crossings

2n+1 petals

[Adams et al]

Crossings V Petals



Twist knots
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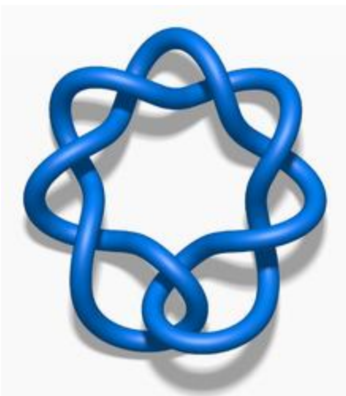
2n+1 petals $\Rightarrow$ crossings < n^2 [Adams et al]

Crossings V Petals



Twist knots
n crossings
n+2 or n+3 petals

Torus(n,n+1)
n^2-1 crossings
2n+1 petals

2n+1 petals $\Rightarrow$ crossings < n^2 [Adams et al]

petals < 2n $\Leftarrow$ n crossings [E Hass Linial Nowik]

Crossings V Petals

Algorithm [E Hass Linial Nowik]

knot diagram — permutation

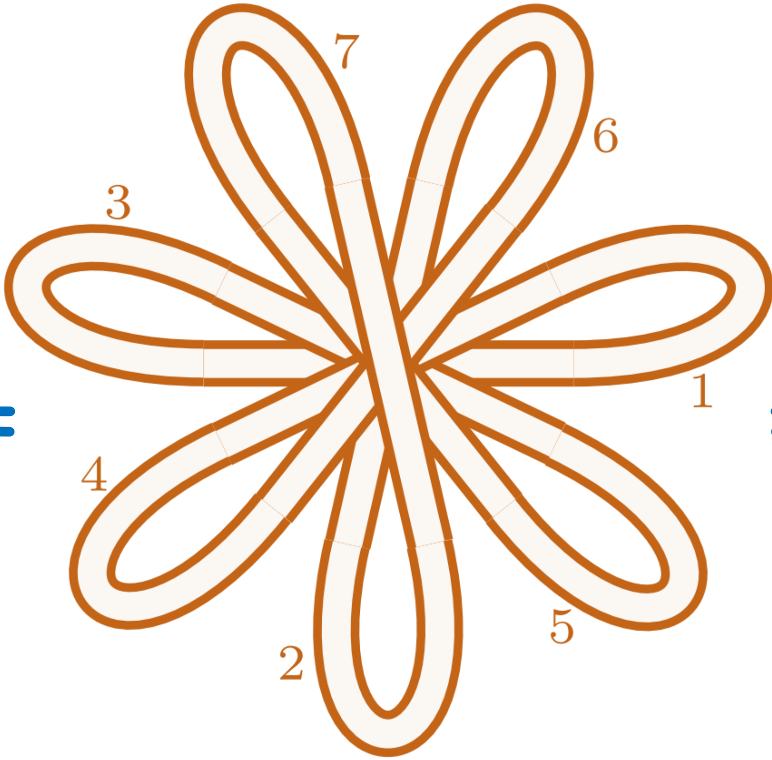
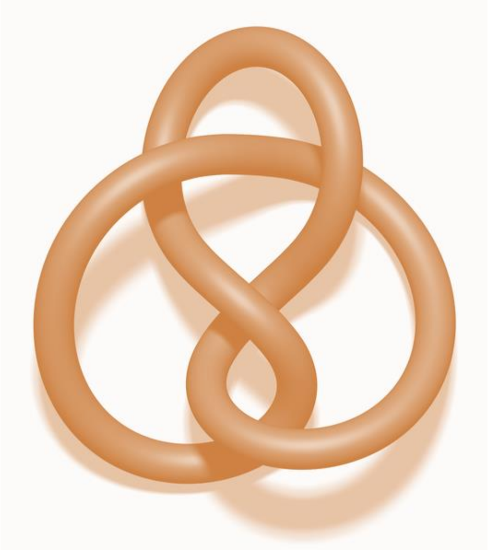
crossings = n ⇒ # petals < 2n

Corollary

There are at least exponentially many n-petal knots.

Notation

K : Permutations -> Knots

K(1357264) =  = 

Invariance to Rotation

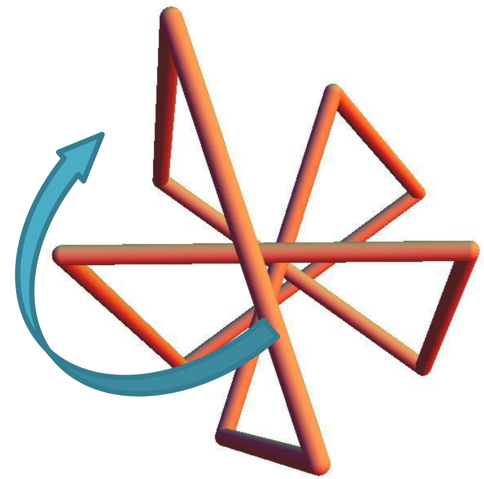
$$K(\rho\cdot\pi) \;=\; K(\pi) \;=\; K(\pi\cdot\rho)$$

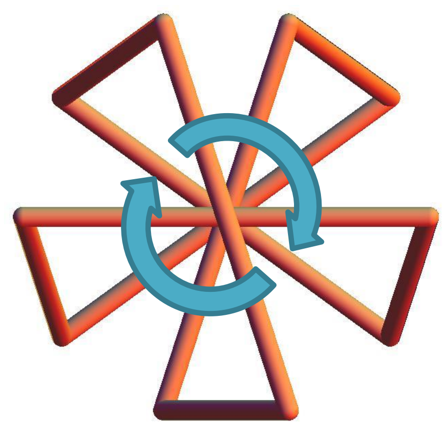
where: $\rho(x) = x+1$

$K(\;2\;8\;4\;1\;7\;9\;5\;3\;6\;) =$

$K(\;1\;7\;3\;9\;6\;8\;4\;2\;5\;) =$

$K(\;7\;3\;9\;6\;8\;4\;2\;5\;1\;)$

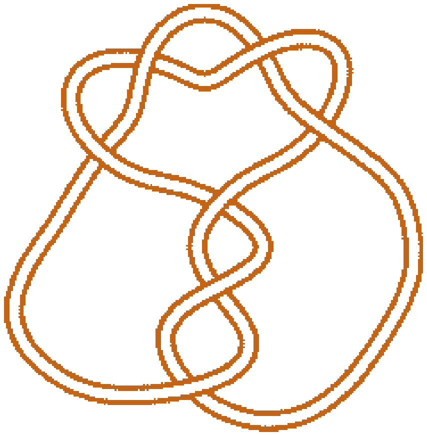




Reflection

K(τ•π) = mirror image of K(π)

where: τ(x) = C-x

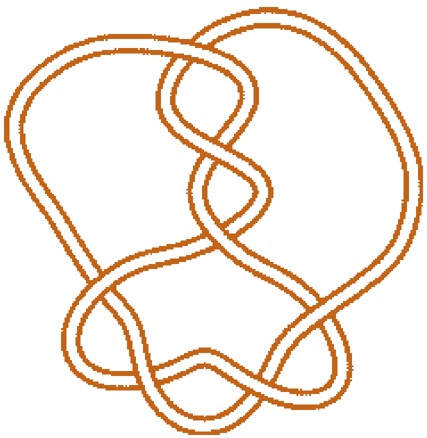


K(π) =

K(2 8 4 1 7 9 5 3 6)

K(τ•π) =

K(8 2 6 9 3 1 5 7 4)

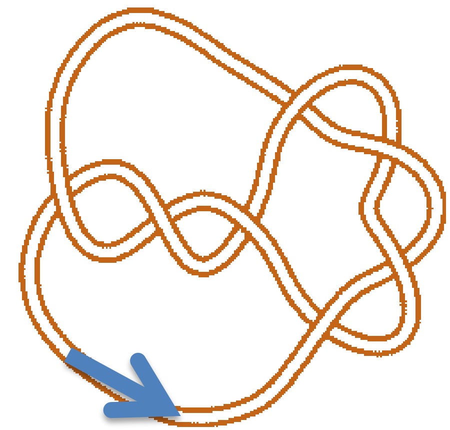


Reverse Orientation

$$K(\pi \cdot \tau) = \text{inverse } K(\tau \cdot \pi)$$

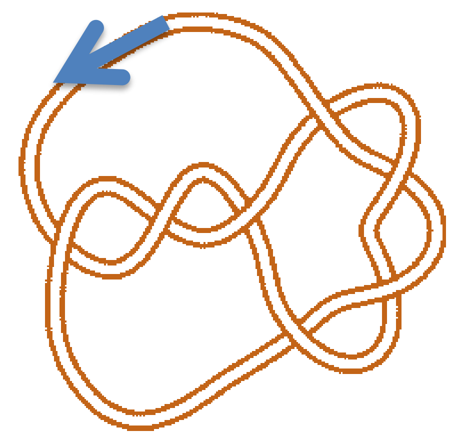
$K(\pi) =$

$K(\ 2\ 8\ 4\ 1\ 7\ 9\ 5\ 3\ 6\ )$

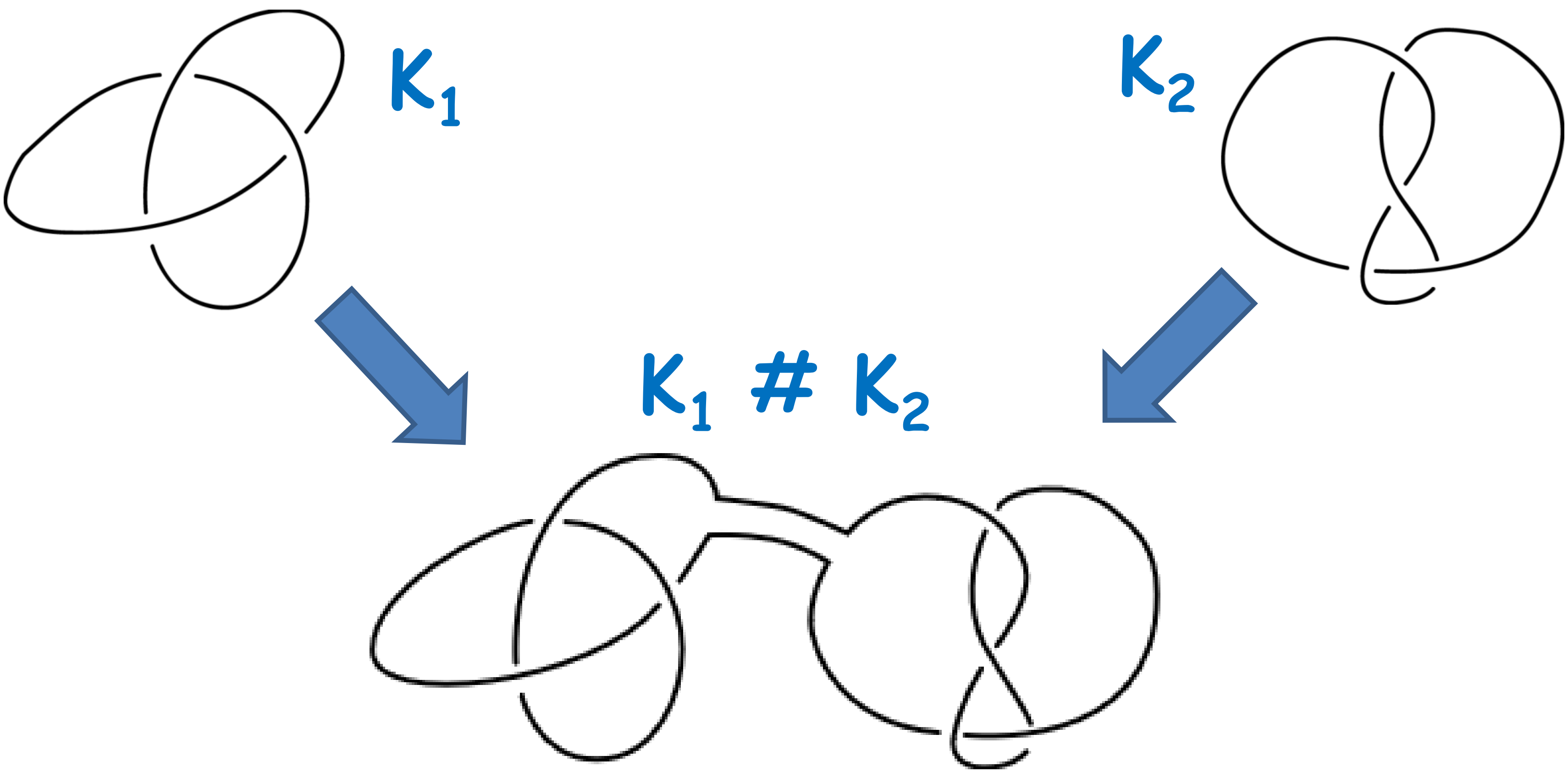


$K(\pi \cdot \tau) =$

$K(\ 6\ 3\ 5\ 9\ 7\ 1\ 4\ 8\ 2\ )$

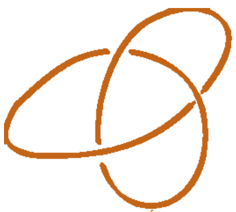


Connected Sum



Connected Sum

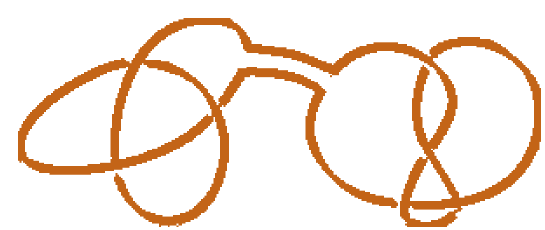
$$K(\pi) \# K(\sigma) = K(\pi \oplus 1 \oplus \sigma)$$

 #  =

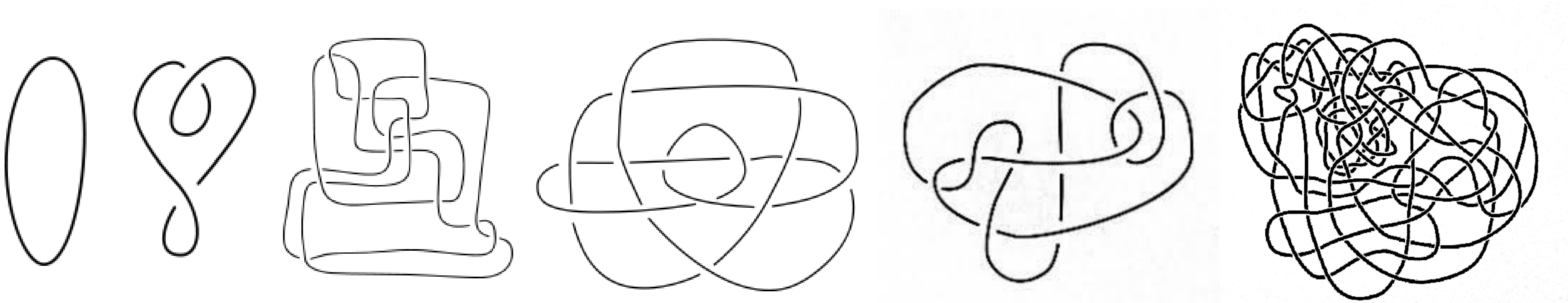
$$K(2\ 4\ 1\ 3\ 5)\ \#\ K(1\ 3\ 5\ 7\ 2\ 6\ 4) =$$

$$K(2\ 4\ 1\ 3\ 5 \oplus 1 \oplus 1\ 3\ 5\ 7\ 2\ 6\ 4) =$$

$$K(2\ 4\ 1\ 3\ 5\ 6\ 7\ 9\ 11\ 13\ 8\ 12\ 10) =$$

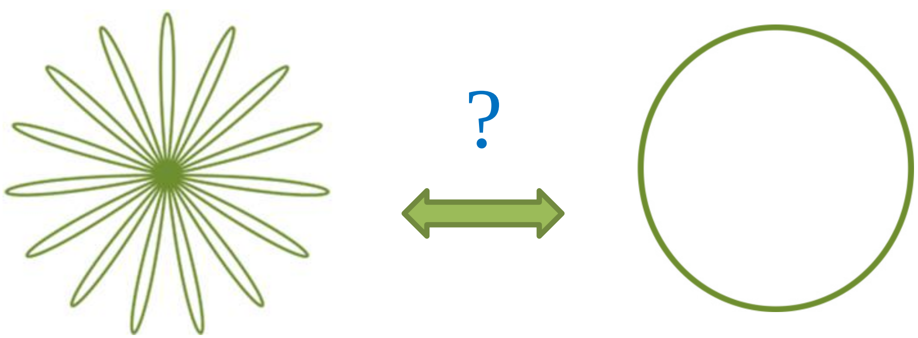


The Unknot



Question

How many permutations represent the unknot?



Counting Unknots

Theorem [E Hass Linial Nowik]

Consider a random $K = K(\pi)$, where $\pi \in S_n$ is uniformly distributed.

$$1/n!! \leq P[K=\text{unknot}] \leq C/n^{0.1}$$

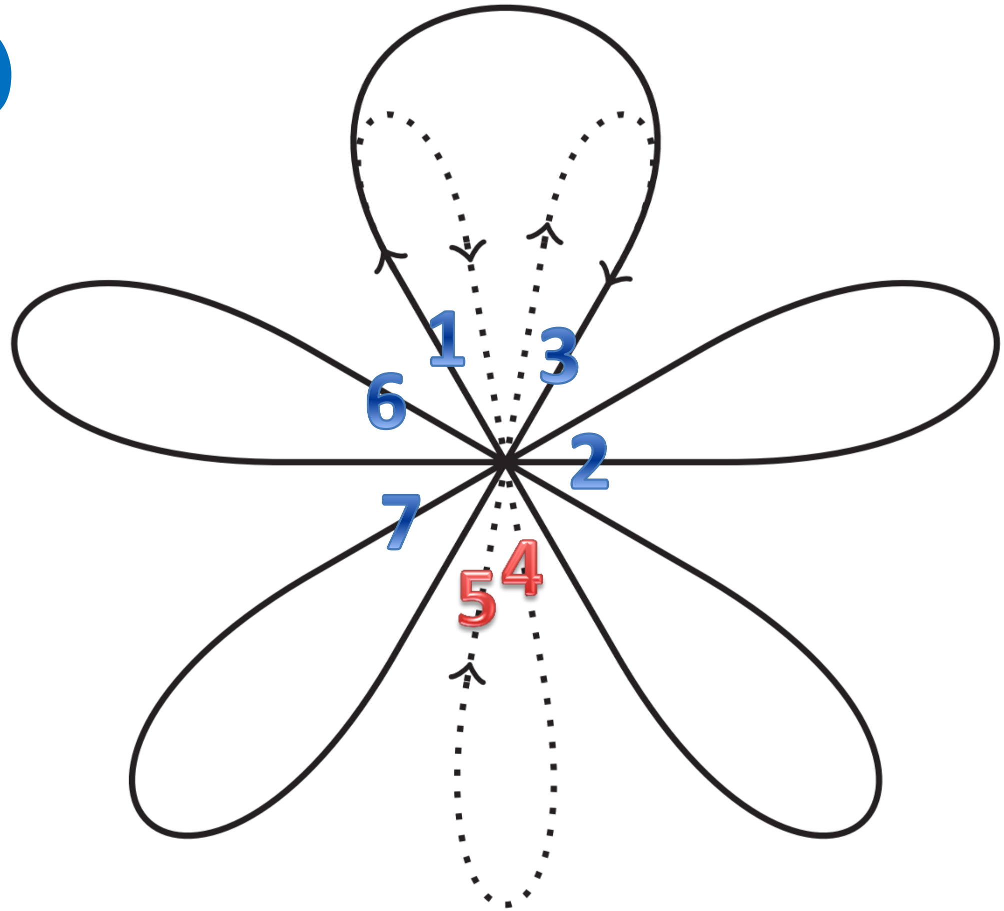
cancellations invariants

Cancellation Moves

K(2 6 1 4 5 3 7)

2 6 1 3 7

= K(2 4 1 3 5)

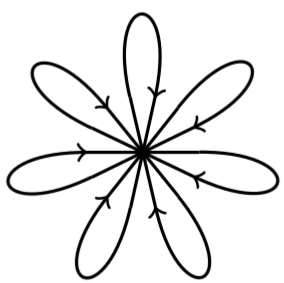


Cancellation Moves

Q. How many permutations of order **2n+1** are **cancellable**?

Example

5246731



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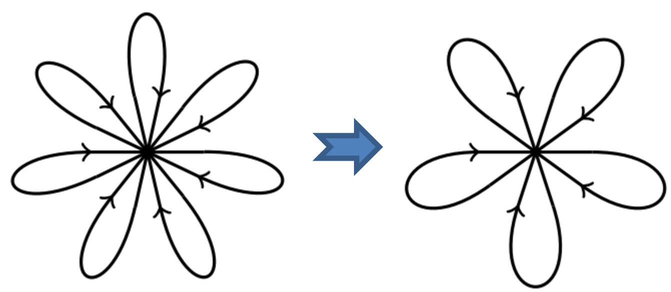
524--31



Cancellation Moves

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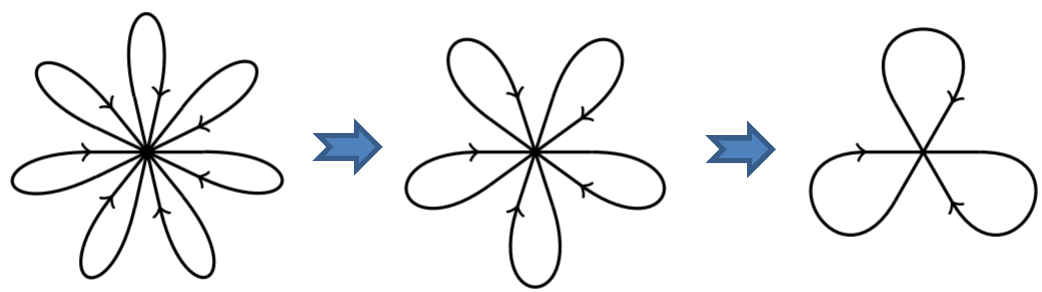
Example

5246731

524--31

52----1



Cancellation Moves

Q. How many permutations of order **2n+1** are **cancellable**?

Example

5246731

524--31

52----1

------1

Cancellation Moves

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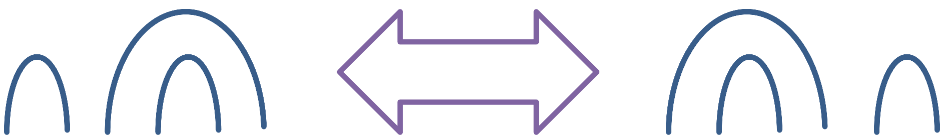


5246731　　　　**1234567**

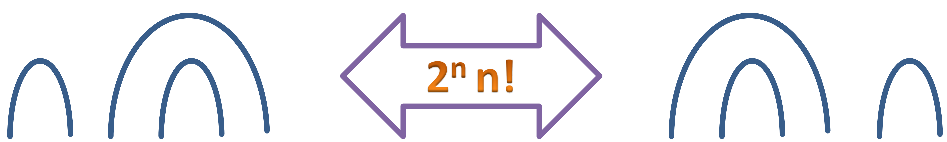
524--31

52----1

------1

Cancellation Moves

Q. How many permutations of order $2n+1$ are **cancellable**?



5246731 1234567

524--31

52----1

------1

A. Between $8^n \cdot n!$ and $32^n \cdot n!$

*up to **poly(n)**

A Culprit Knot

Q. Is every permutation in K^{-1}(**unknot**) **cancellable**?

A. Not! [Adams & co.]

K(1 9 3 5 7 10 2 4 8 11 6)

= unknot

Knot Invariants

I : Knots -> Any set

Three-Dimensional Invariants.			
☐ Arc Index	☐ Braid Index	☐ Braid Length	☐ Bridge Index
☐ Crosscap Number	☐ Crossing Number	☐ Determinant	☐ Morse-Novikov Number
☐ Nakanishi Index	☐ Polygon Index	☐ Seifert Matrix	☐ Small or Large
☐ Super Bridge Index	☐ Symmetry Type	☐ Three Genus	☐ Thurston-Bennequin Number
☐ Torsion Numbers	☐ Tunnel Number	☐ Turaev Genus	☐ Unknotting Number
☐ Width			

Concordance and Four-Dimensional Invariants.			
☐ Arf Invariant	☐ Clasp Number	☐ Smooth Concordance Genus	☐ Topological Concordance Genus
☐ Smooth Concordance Order	☐ Topological Concordance Order	☐ Algebraic Concordance Order	☐ Smooth Four Genus
☐ Topological Four Genus	☐ Smooth 4D Crosscap Number	☐ Topological 4D Crosscap Number	☐ Rasmussen Invariant
☐ Ozsvath-Szabo Tau-Invariant	☐ Signature	☐ Signature function	☐ Smooth Concordance Crosscap Nun
☐ Topological Concordance Crosscap Number			

Positivity.			
☐ Positive Braid	☐ Positive	☐ Strongly Quasipositive	☐ Quasipositive
☐ Positive Braid Notation	☐ Postive PD-Notation	☐ Strongly Quasipostive Braid Notation	☐ Quasipositive Braid Notation

Polynomial Invariants.			
☐ A-Polynomial	☐ Alexander Polynomial	☐ Conway Polynomial	☐ HOMFLY Polynomial
☐ Jones Polynomial	☐ Kauffman Polynomial	☐ Khovanov Polynomial	☐ Khovanov Torsion Polynomial
☐ Show polynomials as coefficient vectors			

Hyperbolic Invariants.			
☐ Volume	☐ Maximum Cusp Volume	☐ Longitude Length	☐ Meridian Length
☐ Longitude Translation	☐ Meridian Translation	☐ Other Short Geodesics	☐ Full Symmetry Group
☐ Chern-Simons Invariant			

http://www.indiana.edu/~knotinfo/

Knot Invariants

I : Knots -> Say, numbers

I ∘ K : Permutations -> Numbers

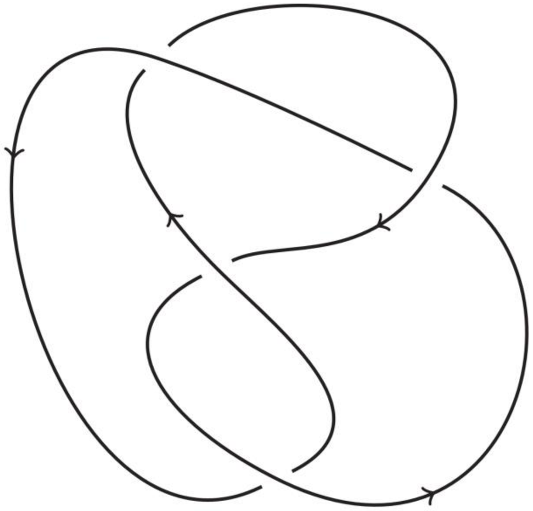
Knot Invariants

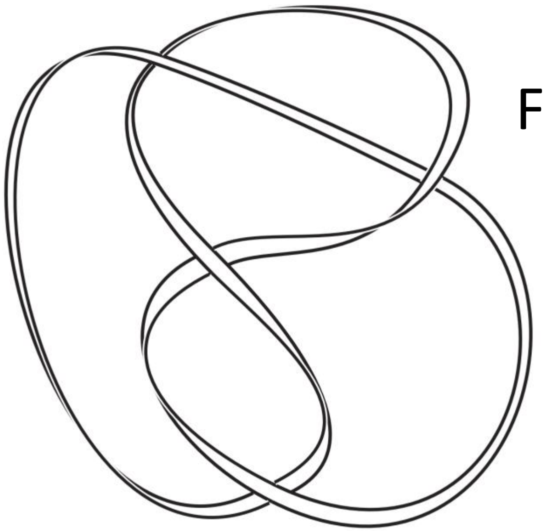
I : Knots -> Say, numbers

$I \circ K$: Permutations -> Numbers

permutation statistics

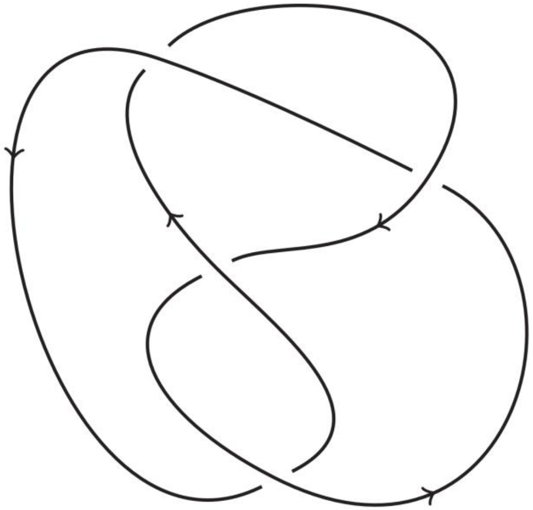
The Framing Number

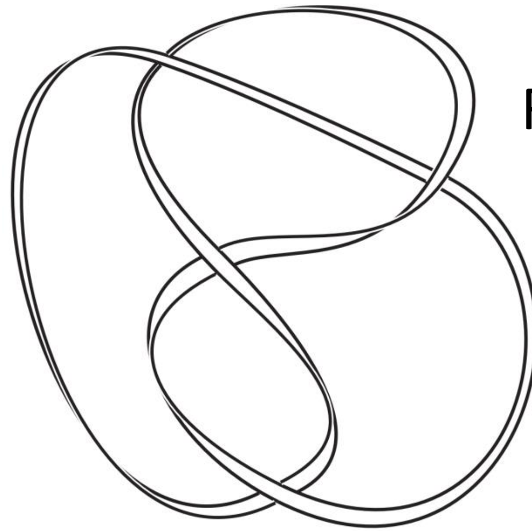




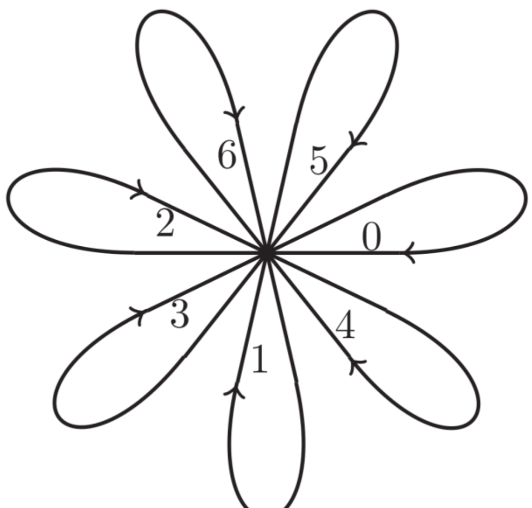
Framed
Knot

The Framing Number

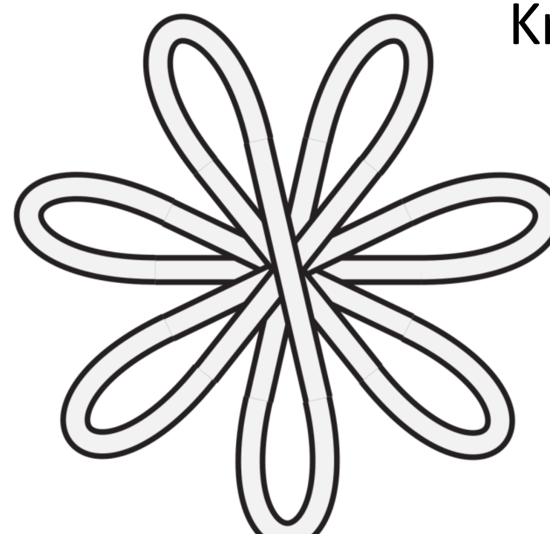




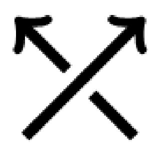
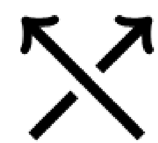
Framed
Knot



Framed
Knot



The Framing Number

$w(K) \;=\; \#$  $-\; \#$ 

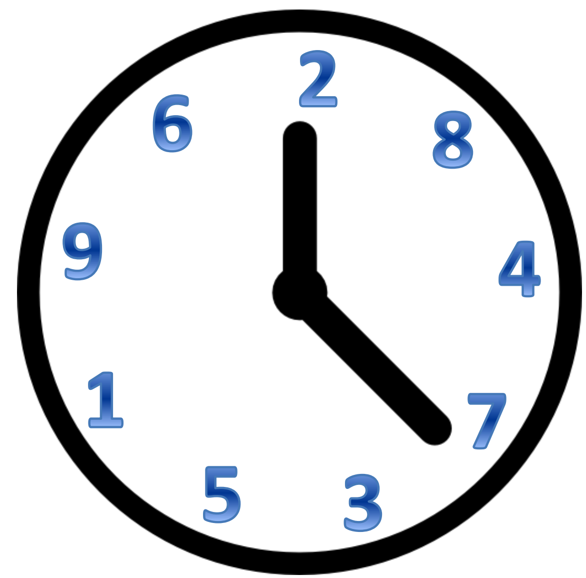
Circular Inversion Number

In terms of

$\pi : \mathbb{Z}_{2n+1} \to \mathbb{R}$

$w : S_n \to \mathbb{Z}$

$$w(\pi) = \sum_{i=0}^{2n} \sum_{j=1}^{n} sign(\pi(i+j) - \pi(i))$$



Framing num / Circular inv

- Antisymmetric $w(\pi \cdot \tau) = w(\tau \cdot \pi) = -w(\pi)$
- Between $-n^2 \leq w(\pi) \leq n^2$
- Computable in time $O(n \log^2 n)$
- Distribution for random $\pi \in S_{2n+1}$

$$\frac{w(K_{2n+1})}{n} \xrightarrow[n\to\infty]{} \mathbf{W} \sim \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{A_k}{k}$$

$A_k \sim$ iid

$f_A(x) = 1 / \pi \cosh x$

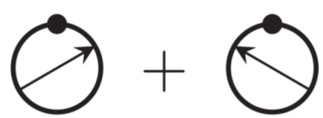
- Equidistributed with alternating inv:

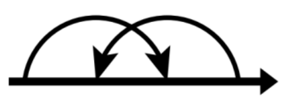
$$\sum_{x<y} (-1)^{x+y} \, sign(\pi(y) - \pi(x))$$

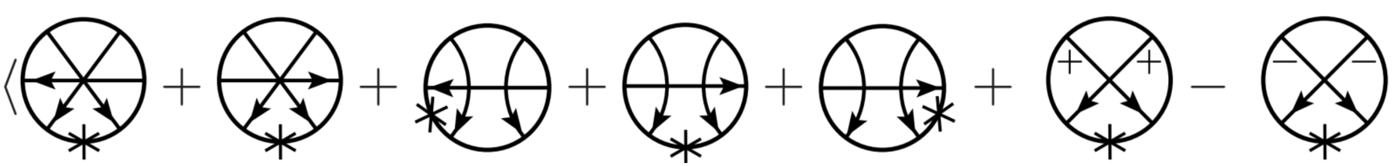
Finite Type Invariants

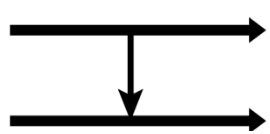
Computed by **Gauss Diagram Formulas**, which represent summations over crossings

[Vassiliev 1990] [Polyak, Viro, Goussarov 1994, 2000]

$w(D) = \Big\langle$  $, D \Big\rangle$

$c_2(K) = \langle$  $, D\rangle$

$j_3(K) = -6\langle$  $, G\rangle$

$lk(K_1, K_2) = \langle$  $, G(L)\rangle$

The Casson Invariant

c_2 = coef of x^2 in the Alexander-Conway Polynomial and modified Jones Polynomial

Proposition: For $\pi \in S_{2n+1}$

$$c_2(\pi) = \sum_{abcd^*} (-1)^{a+b+c+d+1}/24$$

*over $a,b,c,d \in \{1,\ldots,2n+1\}$ where

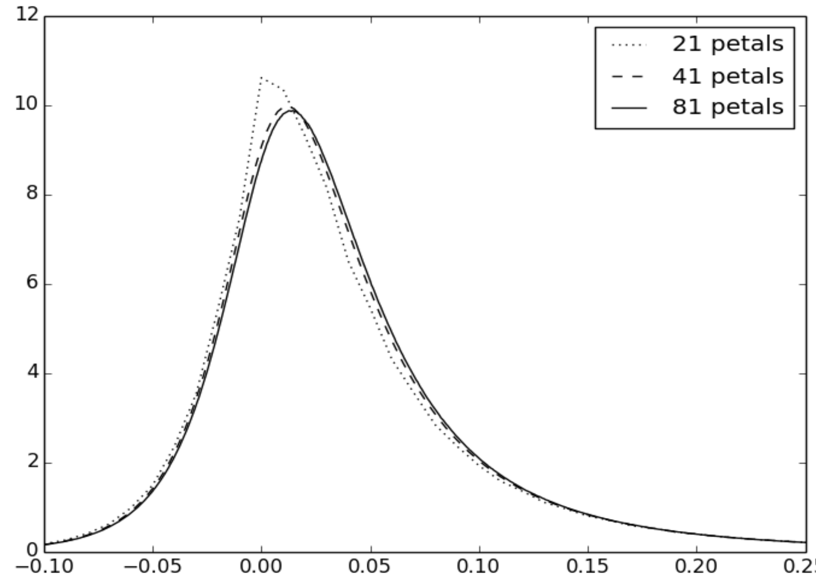
$$\pi(1^{st}) < \pi(3^{rd}) \quad \text{and} \quad \pi(2^{nd}) > \pi(4^{th})$$

w.r.t. the sorted multi-set $\{a,b,c,d\}$

The Casson Invariant

Distribution for

random $\pi \in S_{2n+1}$



$$E[c_2] = \frac{n^2 - n}{24}$$

$$V[c_2] = \frac{8n^4 + 2n^3 - 7n^2 - 3n}{1440}$$

$$E\left[c_2^k\right] = \mu_k n^{2k} + O(n^{2k-1})$$

Conjecture [E Hass Linial Nowik]

Let v_m be a finite type invariant of order m.

For random $\pi \in S_{2n+1}$, $v_m(K_{2n+1})/n^m$ weakly converges to a limit distribution as $n \to \infty$.

THANK YOU!