Universal Permutations

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Permutation Patterns 2018
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Universality

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$$\frac{n^2}{e^2} \leqslant L_n \leqslant n^2.$$

First results

Theorem (Eriksson, Eriksson, Linusson, Wästlund; 2007).

$$L_n \leqslant \frac{2}{3}n^2 + O\left(n^{3/2}(\log n)^{1/2}\right).$$

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Theorem (Miller; 2009).

$$L_n \leqslant \frac{1}{2}n^2 + \frac{1}{2}n.$$

Infinite zigzag word

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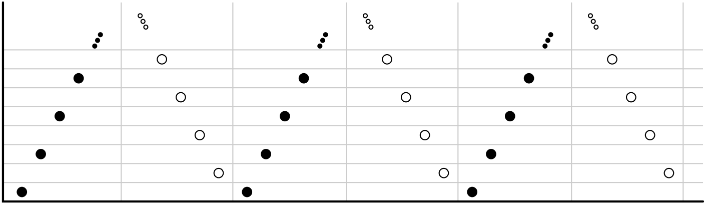
$$Z = (1\ 3\ 5\ 7 \cdots)(\cdots 8\ 6\ 4\ 2)(1\ 3\ 5\ 7 \cdots)(\cdots 8\ 6\ 4\ 2)(1\ 3\ 5\ 7 \cdots)(\cdots 8\ 6\ 4\ 2) \cdots$$

Infinite zigzag word



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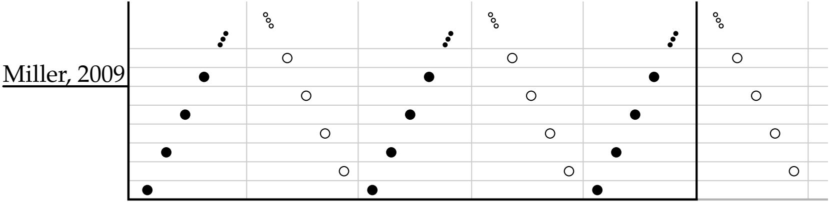

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Theorem (Miller; 2009).

For all $\pi \in S_n$, either π or π^{+1} embed into the first n runs of Z.

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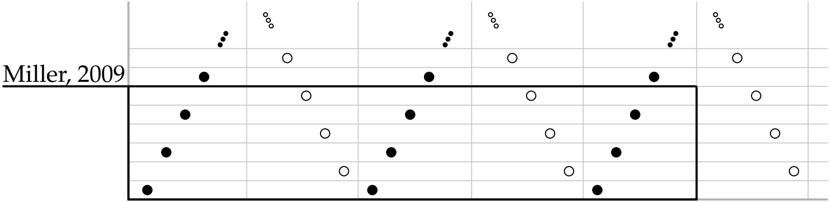

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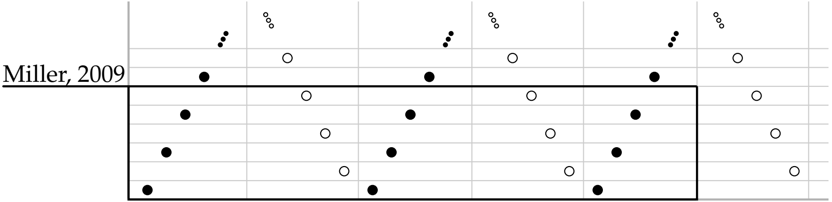

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Corollary.

Let z_n be the restriction of Z to include the first n runs and $n+1$ values,

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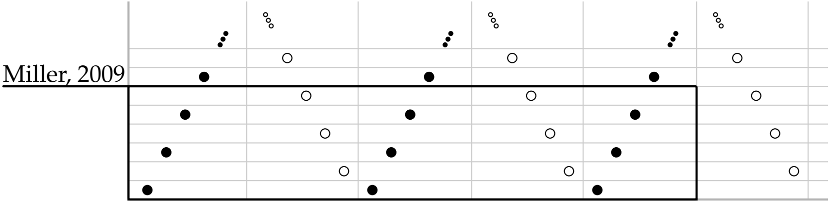

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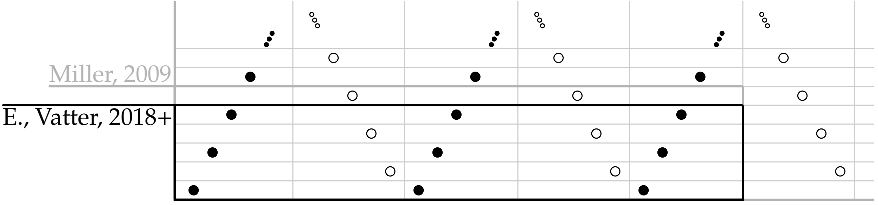
Theorem (Miller; 2009).

For all $\pi \in S_n$, either π or π^{+1} embed into the first n runs of Z.

Corollary.

Let z_n be the restriction of Z to include the first n runs and $n+1$ values, and let π be a permutation formed from z_n by breaking ties between values arbitrarily. Then π is n-universal.

Infinite zigzag word



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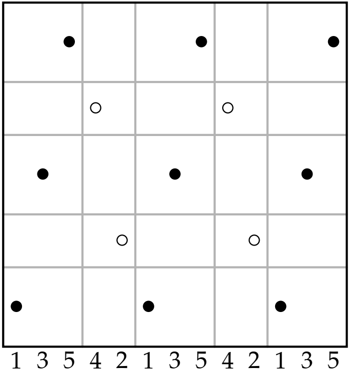
Theorem (E., Vatter; 2018+).

Let z_n^* be the restriction of Z to include the first n runs and n values, and let π be a permutation formed from z_n by breaking ties between values in a decreasing fashion. Then π is almost n-universal.

Construction and state of affairs



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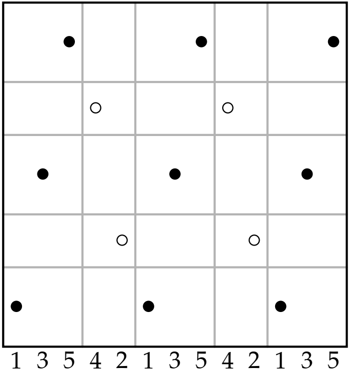

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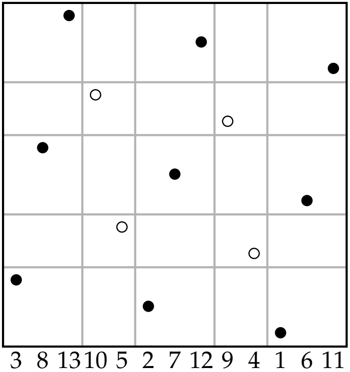

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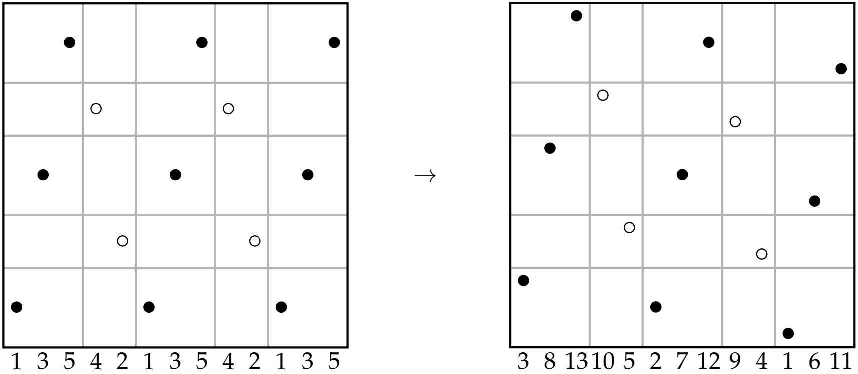

		n	1	2	3	4	5
(Miller)	$(n^2+n)/2$		1	3	6	10	15
(actual)	L_n		1	3	5	9	13

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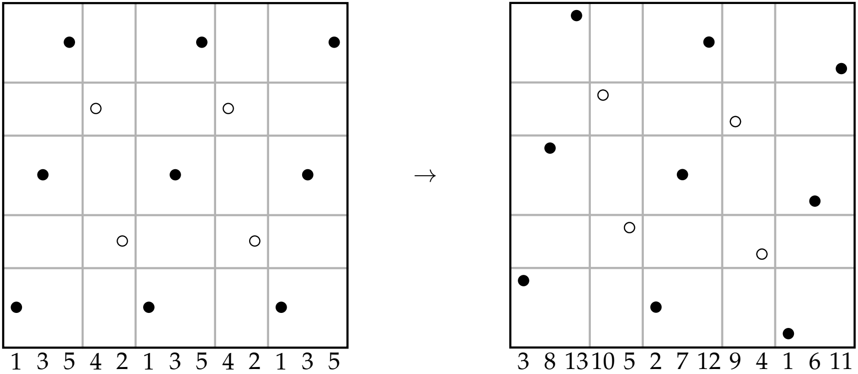

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(actual)	L_n	1	3	5	9	13

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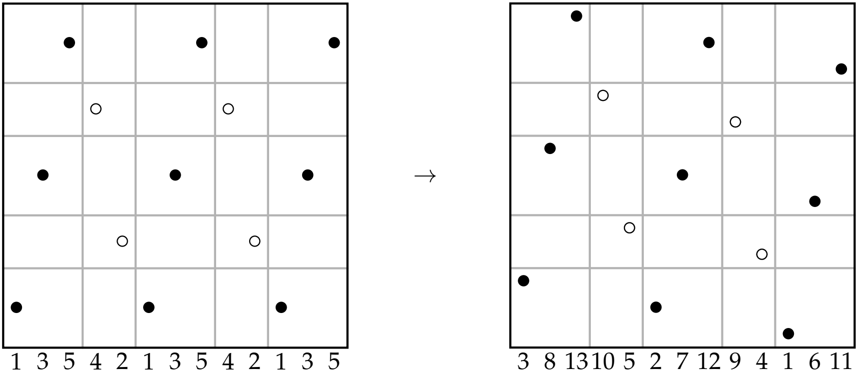

	n	1	2	3	4	5	6
(Miller)	$(n^2+n)/2$	1	3	6	10	15	21
(E., Vatter)	$\lceil (n^2+1)/2 \rceil$	1	3	5	9	13	19
(actual)	L_n	1	3	5	9	13	

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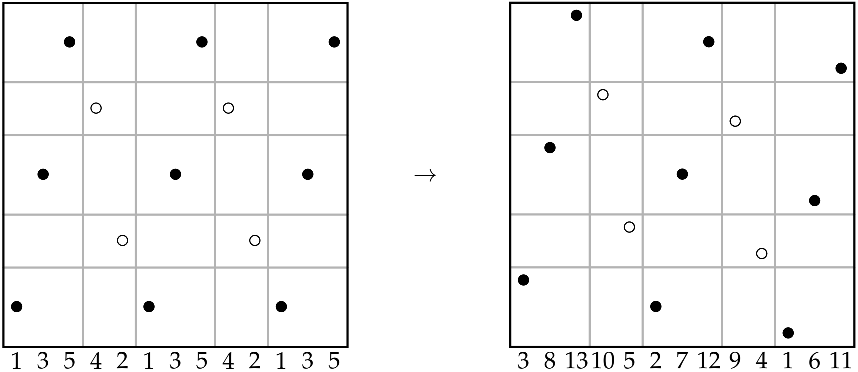

	n	1	2	3	4	5	6
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(actual)	L_n	1	3	5	9	13	$\leqslant 18$

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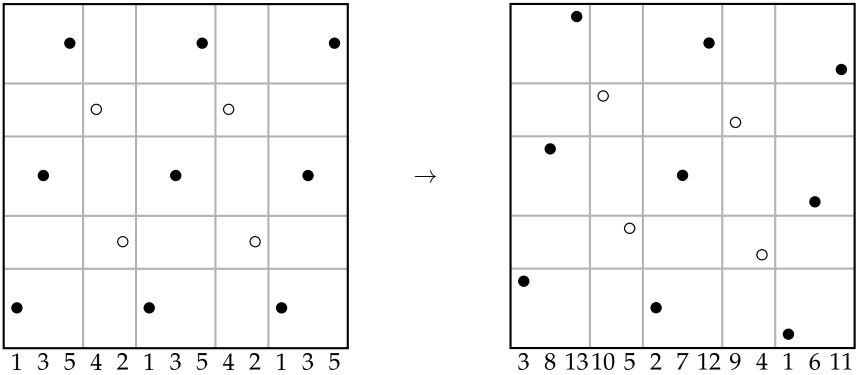

	n	1	2	3	4	5	6	7
(Miller)	$(n^2+n)/2$	1	3	6	10	15	21	28
(E., Vatter)	$\lceil (n^2+1)/2 \rceil$	1	3	5	9	13	19	25
(actual)	L_n	1	3	5	9	13	$\leqslant 18$	$\leqslant 24$

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- Given a class $\mathcal{C}$, determine the length of the shortest n-universal permutations for $\mathcal{C}$ which themselves lie in $\mathcal{C}$.

Universal permutations for $\mathcal{L}$

Let $\mathcal{L} = \text{Av}(231, 312)$ be the class of layered permutations.

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Proposition (Albert, E., Pantone, and Vatter; 2018).

Given any permutation π, there is a layered permutation of the same length that contains every layered permutation contained in π.

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Proposition (Albert, E., Pantone, and Vatter; 2018).

Given any permutation π, there is a layered permutation of the same length that contains every layered permutation contained in π.

Corollary (Albert, E., Pantone, and Vatter; 2018).

Among all shortest permutations which are n-universal for $\mathcal{L}$, there is one which itself is layered.

Layerization

Proof of Proposition.

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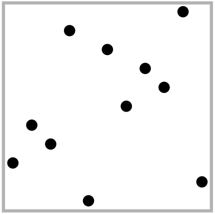

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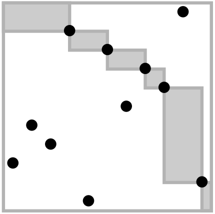

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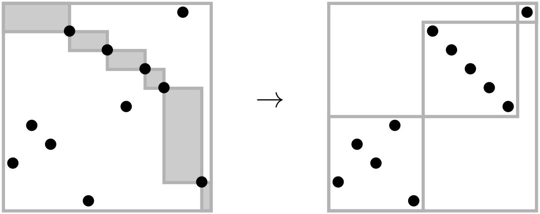

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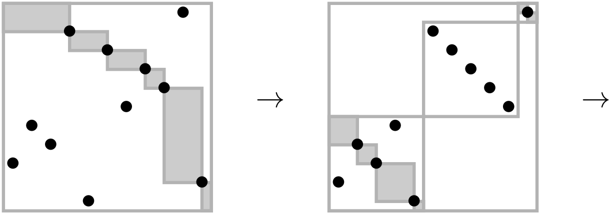

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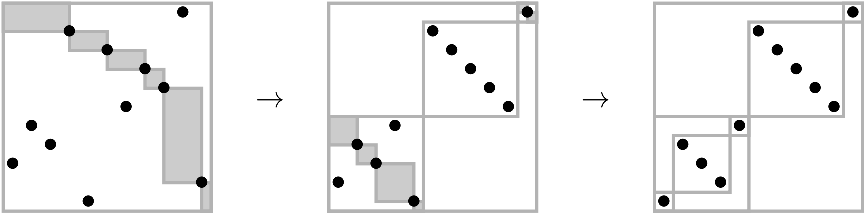

□

Theorem (Albert, E., Pantone, and Vatter; 2018).
For all n, the length of the shortest permutation that is n-universal for layered permutations is given by the sequence defined by

$$a(n) = n + \min\{a(k) + a(n-k-1) \; : \; 0 \leqslant k \leqslant n-1\}$$

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Note: As a consequence, $a(n) \sim n\log_2(n)$.

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Thank you!