Local convergence for random permutations

The case of uniform pattern-avoiding permutations

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Dartmouth College, Hanover, New Hampshire

Our goal

Study limits of random permutations when the size tends to infinity

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1. Scaling limits:
   Limiting objects: Permutons
   Corresponding statistic: $\widetilde{\mathrm{occ}}(\pi, \sigma)$, for all $\pi \in \mathcal{S}$
   Concrete examples: ρ-avoiding permutations for $|\pi| = 3$, separable permutations, substitution-closed classes, Mallows permutations, ...

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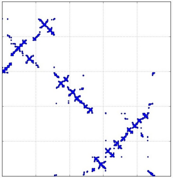

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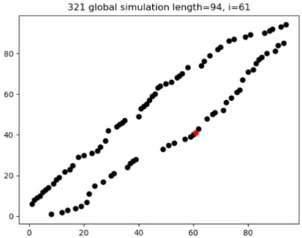
Some simulations



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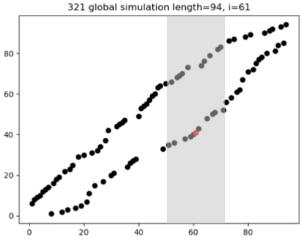

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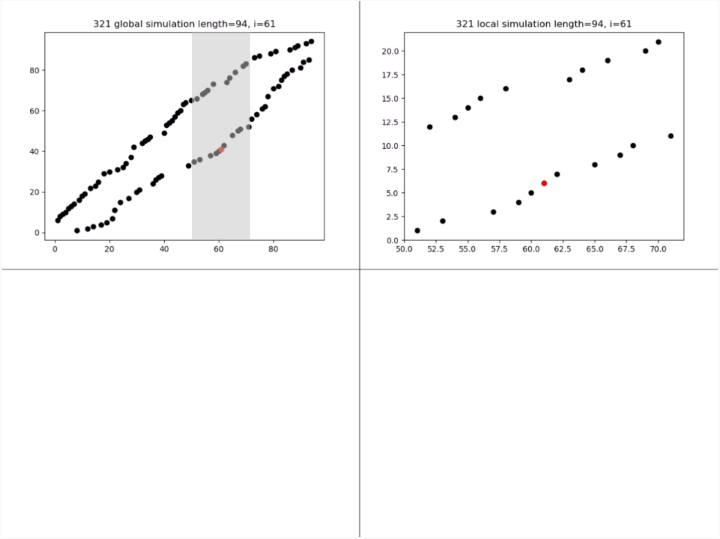

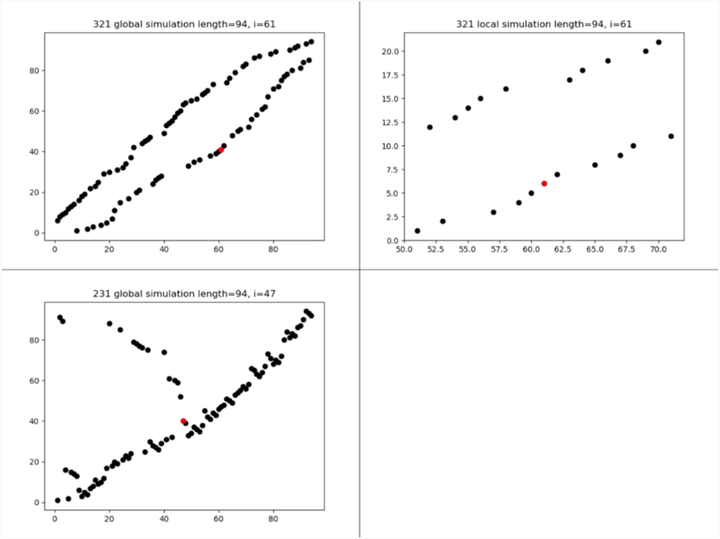
321 global simulation length=94, i=61
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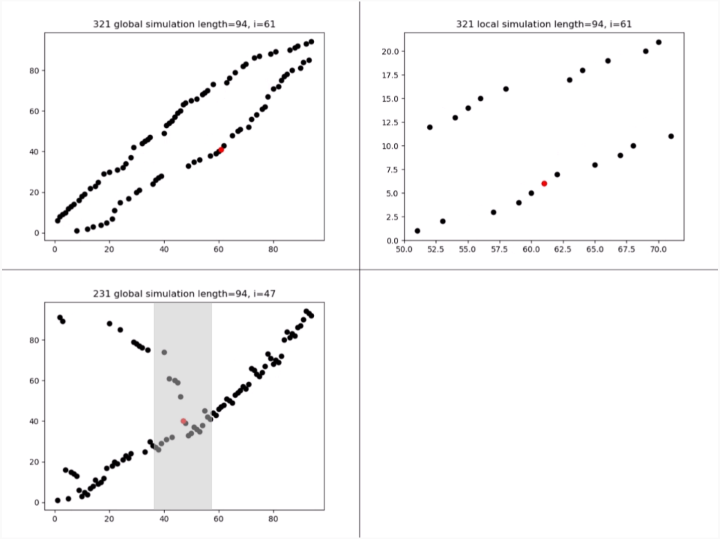

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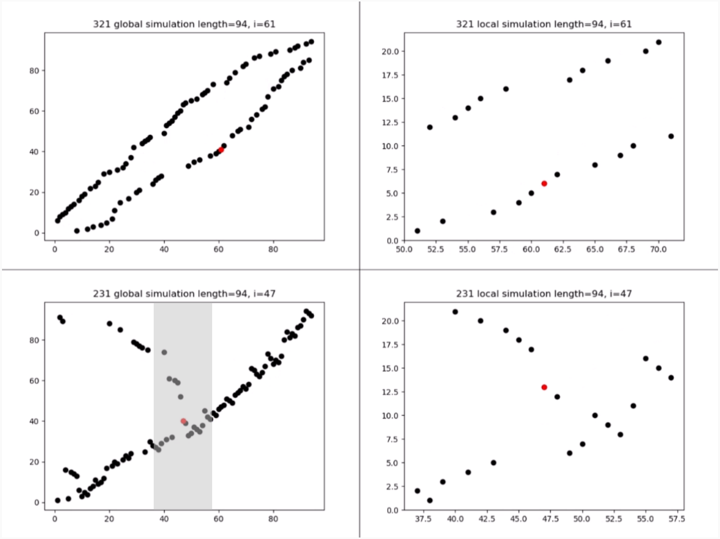

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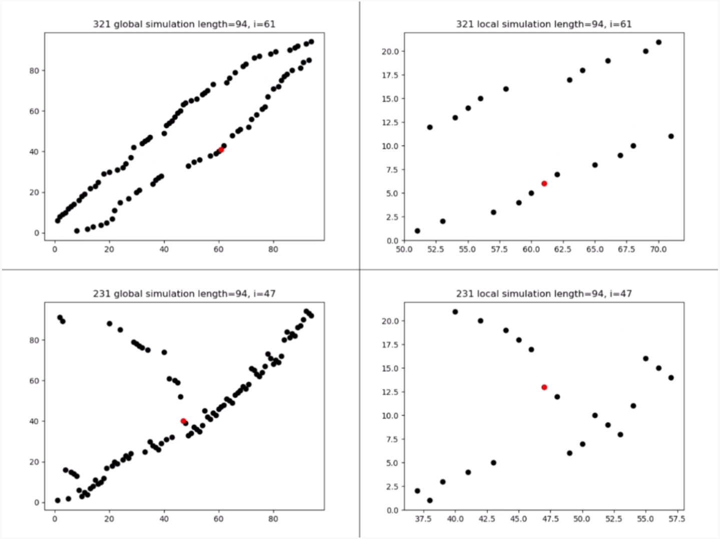
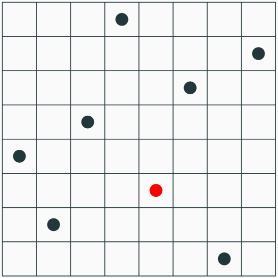


The space of rooted permutations

Finite rooted permutations

$$\sigma = 4\ 2\ 5\ 8\ \textcolor{red}{3}\ 6\ 1\ 7$$



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A finite rooted permutation is a pair (σ, i), where $\sigma \in \mathcal{S}^n$ and $i \in [n]$.
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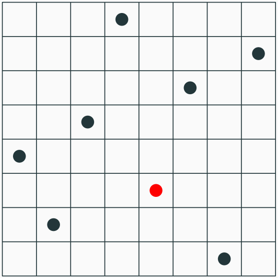

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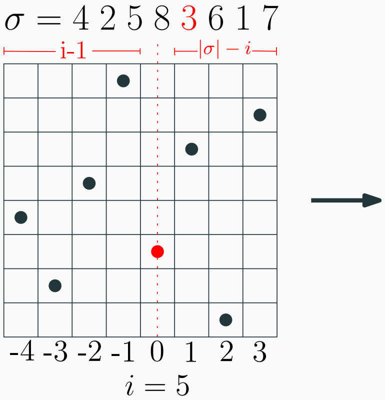

$$A_{\sigma,i} = [-i+1, |\sigma| - i]$$

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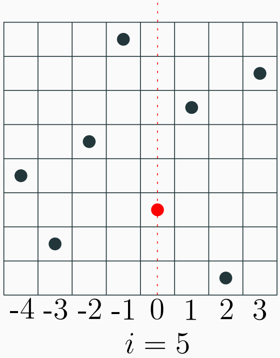

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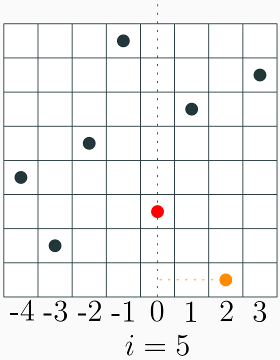

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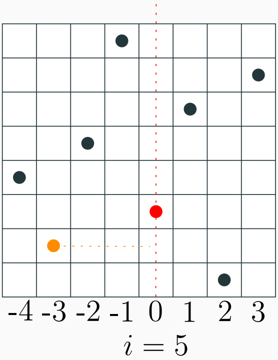

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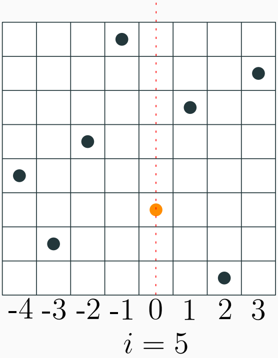

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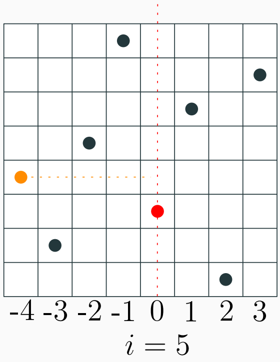

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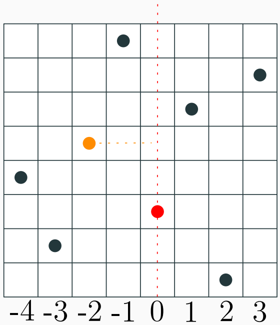

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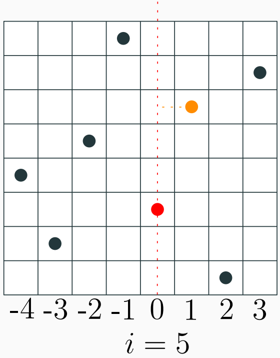

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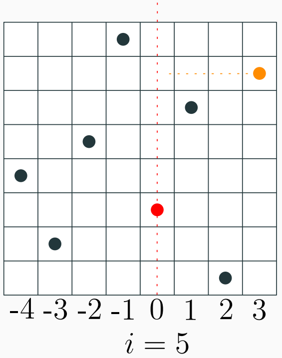

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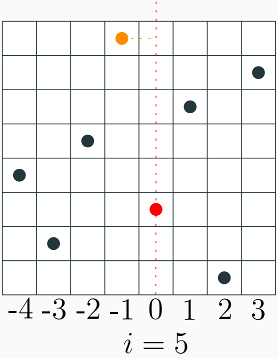

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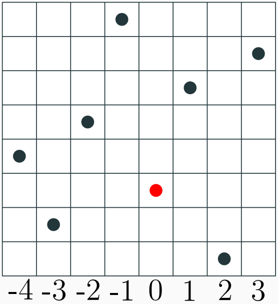

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GOAL: Define a notion of local convergence in $\tilde{\mathcal{S}}_\bullet$ and study limits of random permutations when the size tends to infinity.

Restriction function around the root

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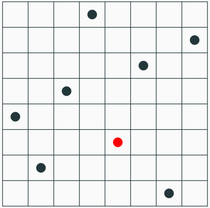

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Definition

The restriction function around the root is defined, for every $h \in \mathbb{N}$, by

$$\begin{aligned} r_h : \quad \tilde{\mathcal{S}}_\bullet &\longrightarrow \qquad \mathcal{S}_\bullet \\ (A, \preccurlyeq) &\mapsto \big(A \cap [-h, h], \preccurlyeq\ \big)\ . \end{aligned}$$

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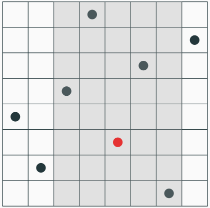

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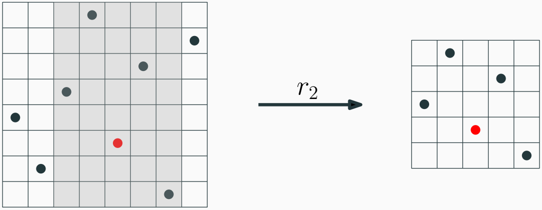

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Local distance for $\tilde{\mathcal{S}}_\bullet$

Definition

We say that a sequence $(A_n, \preccurlyeq_n)_{n\in\mathbb{N}}$ of rooted permutations in $\tilde{\mathcal{S}}_\bullet$ is locally convergent to an element $(A, \preccurlyeq) \in \tilde{\mathcal{S}}_\bullet$, if for all $H > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

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Theorem

The metric space $(\tilde{\mathcal{S}}_\bullet, d)$ is a compact Polish space, *i.e.*, compact, separable and complete. Moreover it contains the space $\mathcal{S}_\bullet$ of finite rooted permutation as a dense subset.

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   Limiting objects: Permutons
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Local convergence: the *consecutive occurrences* characterization

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Observation

In this way, a fixed permutation σ naturally identifies a random variable (σ, i) with values in the set $\mathcal{S}_\bullet$.

Definition

We say that a sequence $(\sigma^n)_{n\in\mathbb{N}}$ of elements in $\mathcal{S}$ Benjamini–Schramm converges to a random rooted permutation $\boldsymbol{\sigma}^\infty$, if

$$(\sigma^n, \boldsymbol{i}_n) \xrightarrow{law} \boldsymbol{\sigma}^\infty, \quad \text{w.r.t. the local distance } d.$$

We write $\sigma^n \xrightarrow{BS} \boldsymbol{\sigma}^\infty$ instead of $(\sigma^n, \boldsymbol{i}_n) \xrightarrow{law} \boldsymbol{\sigma}^\infty$.

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Theorem [B.]

For any $n \in \mathbb{N}$, let σ^n be a permutation of size n. TFAE:

(a) $\sigma^n \xrightarrow{BS} \boldsymbol{\sigma}^\infty$, for some random rooted infinite permutation $\boldsymbol{\sigma}^\infty$.

(b) There exists an infinite vector of non-negative real numbers $(\Delta_\pi)_{\pi\in\mathcal{S}}$ such that

$$\widetilde{c\text{-}occ}(\pi, \sigma^n) \to \Delta_\pi, \quad \text{for all patterns } \pi \in \mathcal{S}.$$

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For any $n \in \mathbb{N}$, let σ^n be a permutation of size n. TFAE:

(a) $\sigma^n \xrightarrow{BS} \boldsymbol{\sigma}^\infty$, for some random rooted infinite permutation $\boldsymbol{\sigma}^\infty$.

(b) There exists an infinite vector of non-negative real numbers $(\Delta_\pi)_{\pi\in\mathcal{S}}$ such that

$$\widetilde{c\text{-}occ}(\pi, \sigma^n) \to \Delta_\pi, \quad \text{for all patterns } \pi \in \mathcal{S}.$$

Link: $\mathbb{P}\big(r_h(\boldsymbol{\sigma}^\infty) = (\pi, h+1)\big) = \Delta_\pi, \quad \text{for all} \quad h \in \mathbb{N}, \text{ all } \pi \in \mathcal{S}^{2h+1}.$

Theorem [B.]

If $(\sigma^n)_{n\in\mathbb{N}}$ is a sequence of deterministic permutations:

$$\text{BS:} \qquad \sigma^n \xrightarrow{\text{BS}} \boldsymbol{\sigma}^\infty \qquad \Longleftrightarrow \qquad \widetilde{c\text{-}occ}(\pi, \sigma^n) \to \Delta_\pi,\ \forall \pi \in \mathcal{S}$$

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If $(\boldsymbol{\sigma}^n)_{n\in\mathbb{N}}$ is a sequence of random permutations:

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Weak-local convergence: deterministic & random case

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$$\text{qBS:} \qquad \boldsymbol{\sigma}^n \xrightarrow{qBS} \boldsymbol{\mu}^\infty \qquad \iff \qquad \left(\widetilde{c\text{-}occ}(\pi,\boldsymbol{\sigma}^n)\right)_{\pi\in\mathcal{S}} \xrightarrow{law} (\boldsymbol{\Lambda}_\pi)_{\pi\in\mathcal{S}}$$

w.r.t. the product topology

Our goal

- Study limits of random permutations when the size tends to infinity
  1. Scaling limits:
     Limiting objects: Permutons
     Corresponding statistic: $\widetilde{\mathrm{occ}}(\pi, \sigma)$, for all $\pi \in \mathcal{S}$
     Concrete examples: separable permutations, substitution-closed classes, ...
  2. Local limits:
     Limiting objects: Rooted permutations
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       Concrete examples: ?

Local limit for uniform 231-avoiding permutations

Definition

For all $n > 0$ we define the following probability distribution on $\mathrm{Av}^n(231)$,

$$P_{231}(\pi) := \frac{2^{|LRMax(\pi)|+|RLMax(\pi)|}}{2^{2|\pi|}}, \quad \text{for all} \quad \pi \in \mathrm{Av}^n(231).$$

Theorem [B.]

Let $\boldsymbol{\sigma}^n$ be a uniform random permutation in $Av^n(231)$ for all $n \in \mathbb{N}$, then

$$\widetilde{c\text{-}occ}(\pi, \boldsymbol{\sigma}^n) \xrightarrow{Prob} P_{231}(\pi), \quad \text{for all} \quad \pi \in Av(231).$$

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Let $\boldsymbol{\sigma}^n$ be a uniform random permutation in $Av^n(231)$ for all $n \in \mathbb{N}$, then

$$\widetilde{c\text{-}occ}(\pi, \boldsymbol{\sigma}^n) \xrightarrow{Prob} P_{231}(\pi), \quad \text{for all} \quad \pi \in Av(231).$$

Corollary

There exists a random infinite rooted permutation $\boldsymbol{\sigma}^\infty_{231}$ such that for all $h \in \mathbb{N}$,

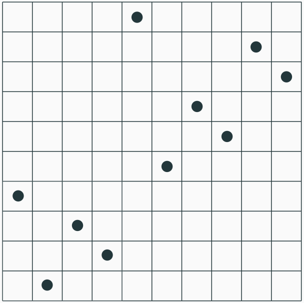
$$\mathbb{P}\big(r_h(\boldsymbol{\sigma}^\infty_{231}) = (\pi, h+1)\big) = P_{231}(\pi), \quad \text{for all} \quad \pi \in \mathcal{S}^{2h+1},$$

and

$$\boldsymbol{\sigma}^n \xrightarrow{qBS} \mathcal{L}(\boldsymbol{\sigma}^\infty_{231}) \quad \text{and} \quad \boldsymbol{\sigma}^n \xrightarrow{aBS} \boldsymbol{\sigma}^\infty_{231}.$$

A bijection between 231-avoiding permutations & binary trees

$\sigma = 4\ 1\ 3\ 2\ 10\ 5\ 7\ 6\ 9\ 8$



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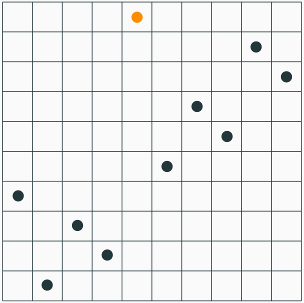

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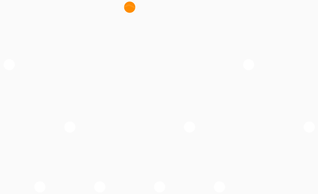

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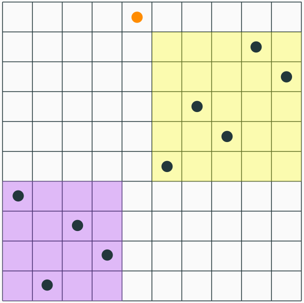

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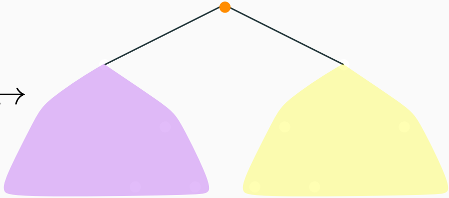

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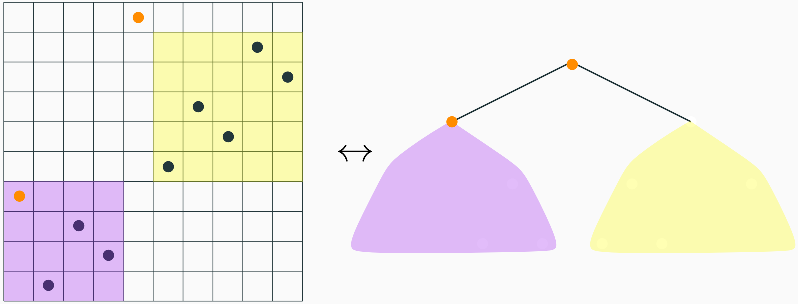

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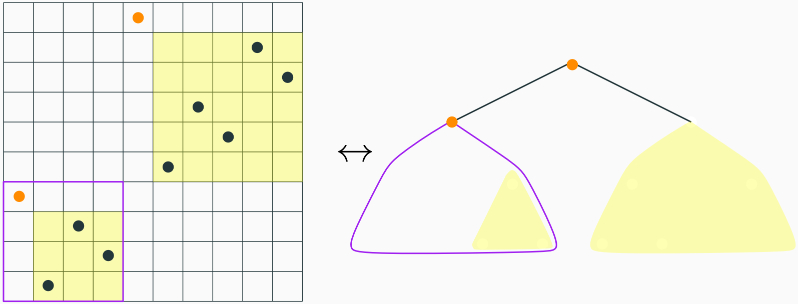

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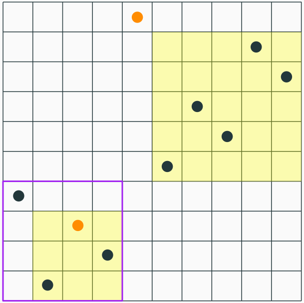

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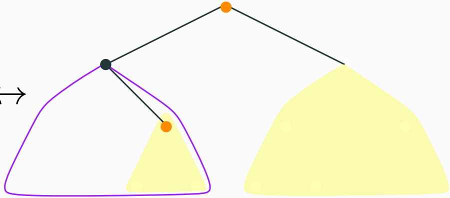

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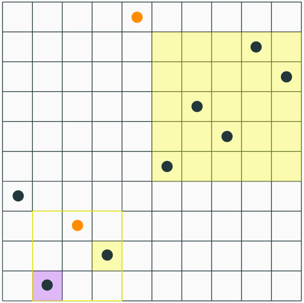

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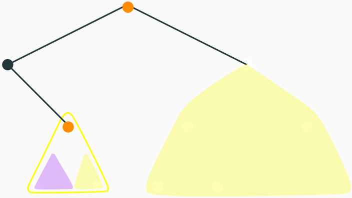

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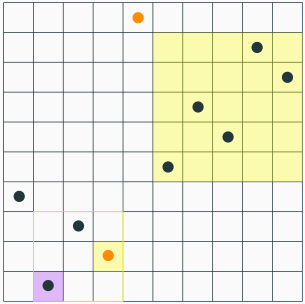

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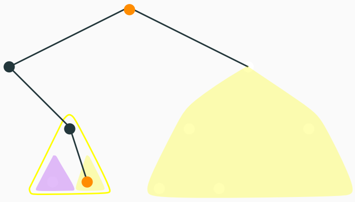

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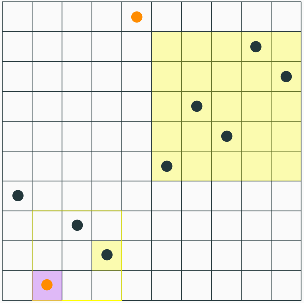

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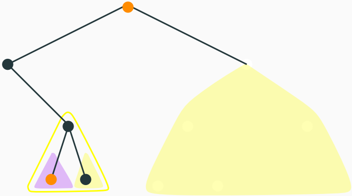

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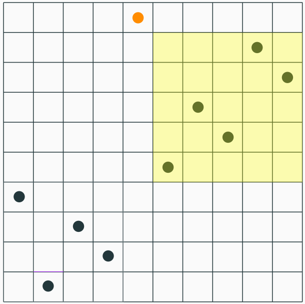

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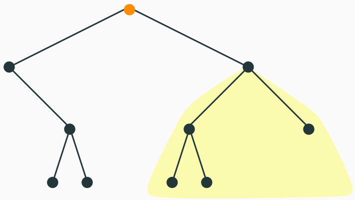

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A binary Galton-Watson tree is a random rooted tree defined as follow:

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    1. We start with the root;
    2. We give to each node children according to an independent copy of $\boldsymbol{\xi}$.

FIRST STEP: Prove that $\mathbb{E}\big[\widetilde{c\text{-}occ}(\pi, \boldsymbol{\sigma}^n)\big] \to P_{231}(\pi),$ for all $\pi \in \text{Av}(231)$

- We relate $\boldsymbol{T}_\delta$ and the sequence $(\boldsymbol{T}_n)_{n\in\mathbb{N}}$ by

$$\begin{aligned}\mathbb{E}\big[F(\boldsymbol{T}_\delta)\big] &= \sum_{n=1}^{+\infty} E\big[F(\boldsymbol{T}_n)\big] \cdot P(|T_\delta| = n)\\ &= \frac{1+\delta}{1-\delta}\sum_{n=1}^{+\infty} \mathbb{E}\big[F(\boldsymbol{T}_n)\big] \cdot C_n \cdot \Big(\frac{1-\delta^2}{4}\Big)^n;\end{aligned}$$

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- Applying singularity analysis and reusing the bijection:

$$\mathbb{E}\left[\widetilde{c\text{-}occ}(\pi, \boldsymbol{\sigma}_n)\right] \to P_{231}(\pi), \quad \text{for all} \quad \pi \in \mathrm{Av}(231).$$

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- Therefore

$$\mathrm{Var}\left(\widetilde{c\text{-}occ}(\pi, \boldsymbol{\sigma}^n)\right) \to 0, \quad \text{for all} \quad \pi \in \mathrm{Av}(231).$$

We finally apply the Second moment method.

Thanks for your attention

Article and slides available at:
http://www.jacopoborga.com
(from midnight also on arXiv)

Questions?

Back-up slides

The construction of the random order σ_{231}^{∞}.

- We consider the following Boltzmann distribution on Av(231) :

$$P(\pi) = \frac{1}{2}\left(\frac{1}{4}\right)^{|\pi|}, \quad \text{for all} \quad \pi \in \mathrm{Av}(231), \qquad P(\emptyset) = \frac{1}{2}.$$

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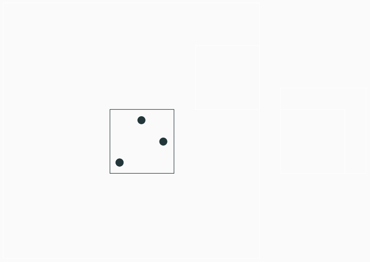

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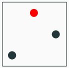

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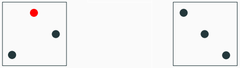

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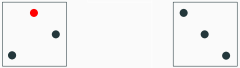

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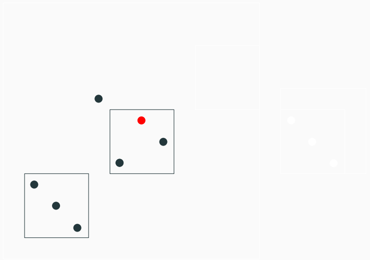

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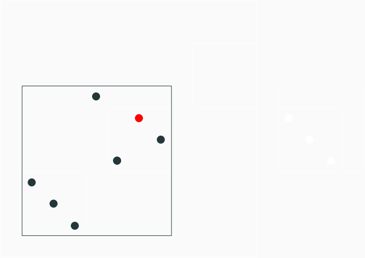

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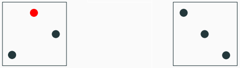

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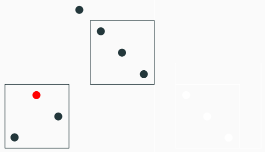

The construction of the random order $\boldsymbol{\sigma}_{231}^{\infty}$.

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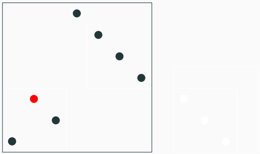

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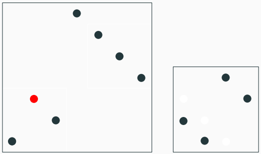

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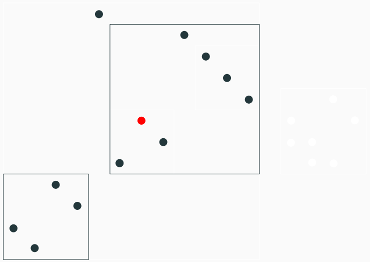

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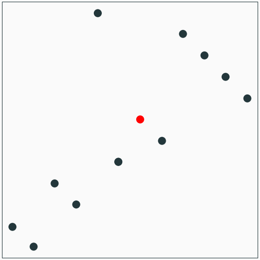

Local limit for uniform 321-avoiding permutations

Definition

For all $n > 0$, we define the following probability distribution on $\mathrm{Av}^n(321)$,

$$P_{321}(\pi) := \begin{cases} \frac{|\pi|+1}{2^{|\pi|}} & \text{if } \pi = 12...|\pi|, \\ \frac{1}{2^{|\pi|}} & \text{if } c\text{-}occ(21, \pi^{-1}) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

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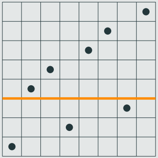
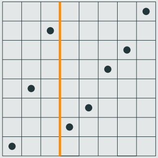
Example

$\pi =$  $\pi^{-1} =$ 

321-AVOIDING PERMUTATIONS

Theorem [B.]

Let $\boldsymbol{\sigma}^n$ be a uniform random permutation in $Av^n(321)$ for all $n \in \mathbb{N}$, then

$$\widetilde{c\text{-}occ}(\pi, \boldsymbol{\sigma}^n) \xrightarrow{Prob} P_{321}(\pi), \quad \text{for all} \quad \pi \in Av(321).$$

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Since the limiting objects $\big(P_{321}(\pi)\big)_{\pi \in Av(231)}$ are deterministic:

Corollary

There exists a random infinite rooted permutation $\boldsymbol{\sigma}^\infty_{321}$ such that for all $h \in \mathbb{N}$,

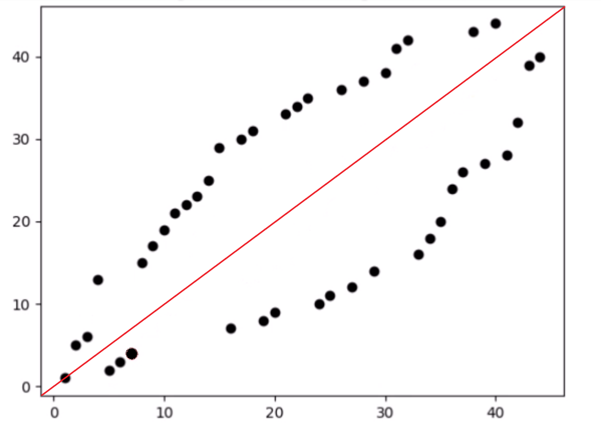
$$\mathbb{P}\big(r_h(\boldsymbol{\sigma}^\infty_{321}) = (\pi, h+1)\big) = P_{321}(\pi), \quad \text{for all} \quad \pi \in \mathcal{S}^{2h+1},$$

and

$$\boldsymbol{\sigma}^n \xrightarrow{qBS} \mathcal{L}(\boldsymbol{\sigma}^\infty_{321}) \quad \text{and} \quad \boldsymbol{\sigma}^n \xrightarrow{aBS} \boldsymbol{\sigma}^\infty_{321}.$$

A bijection between 321-avoiding permutations & trees

It is well known that 321-avoiding permutations can be broken into two increasing subsequences, the first above the diagonal and the second below the diagonal:



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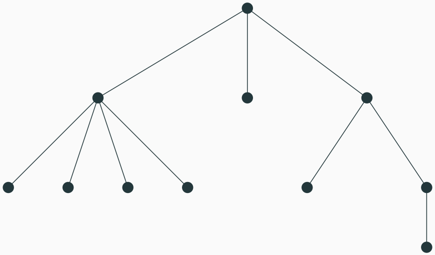

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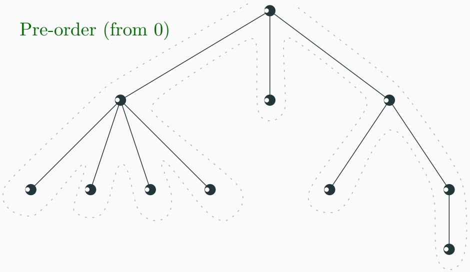

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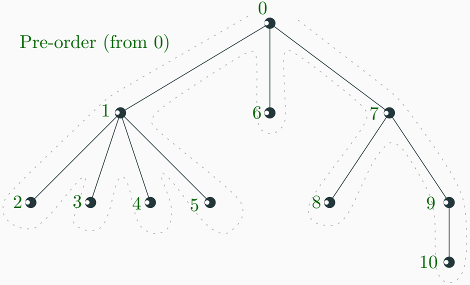

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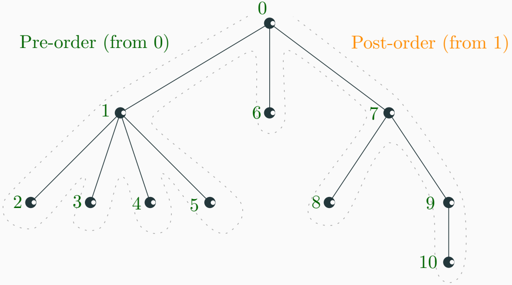

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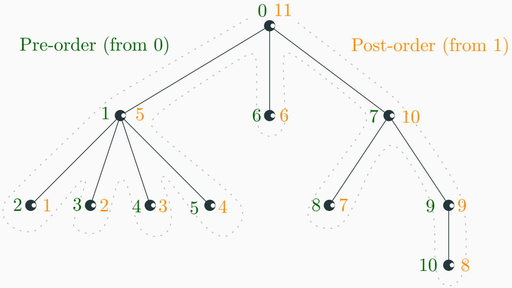

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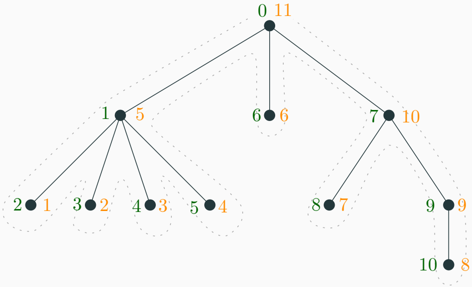

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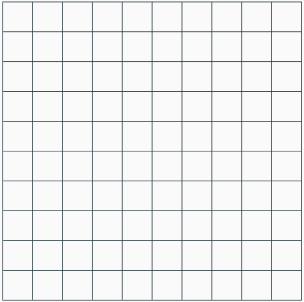

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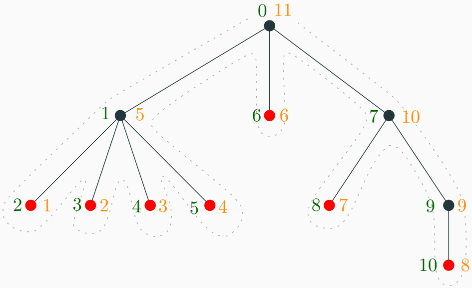

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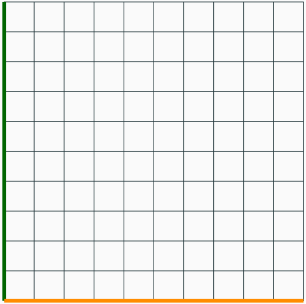

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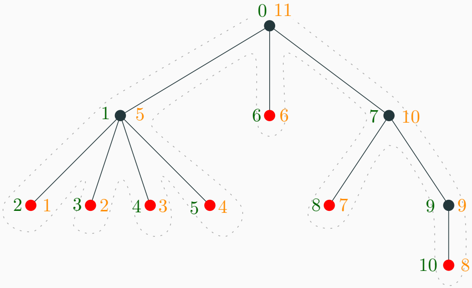

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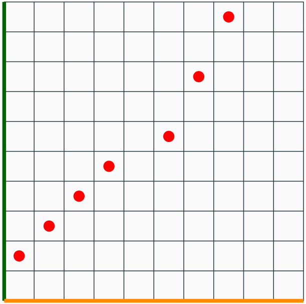

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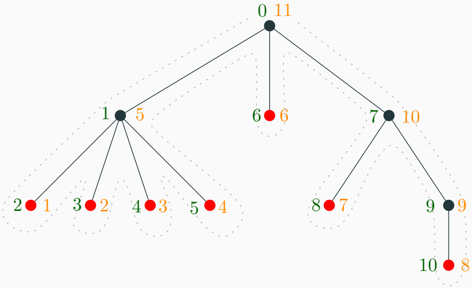

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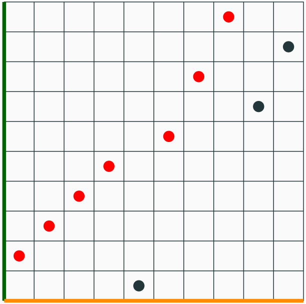

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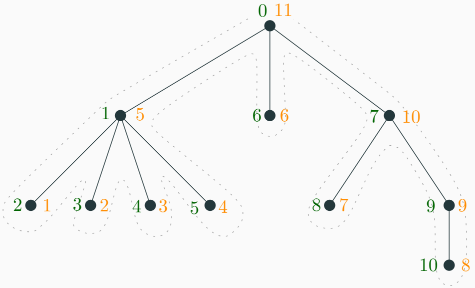

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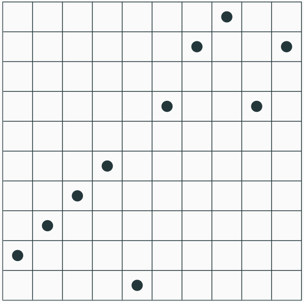

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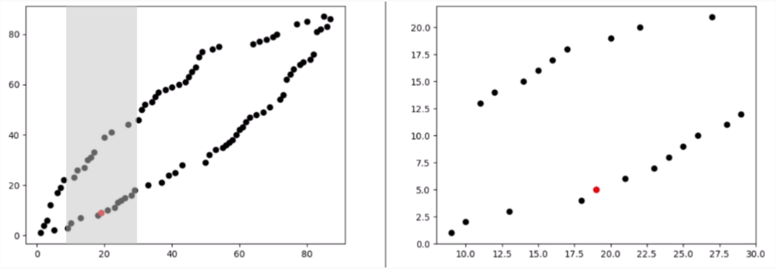

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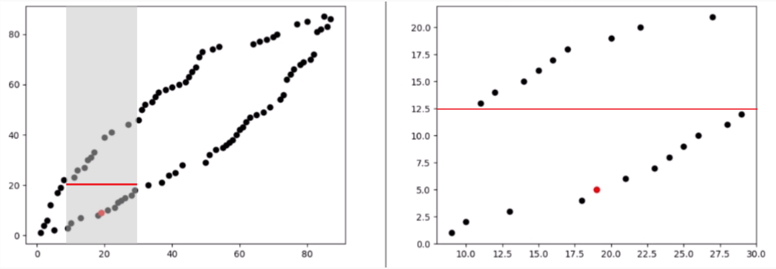

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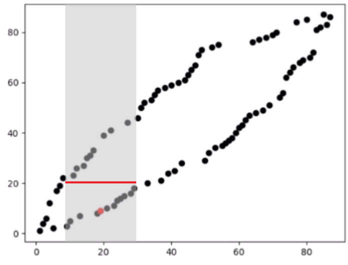
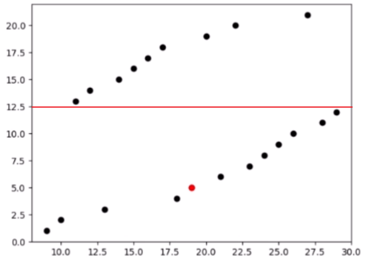
- [Hoffman, Rizzolo, Slivken]: The distance from each subsequence to the diagonal is of order $\sqrt{n} \cdot \boldsymbol{e}$;





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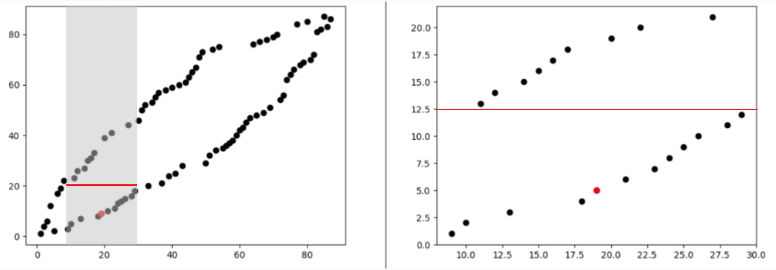

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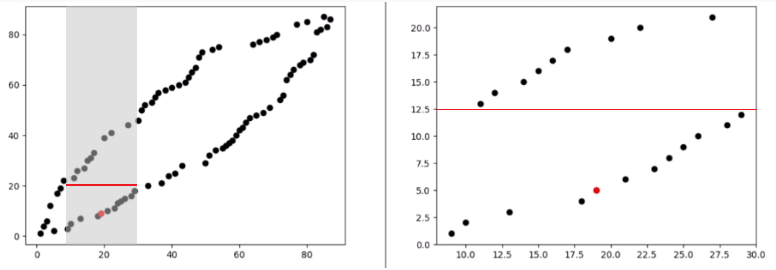

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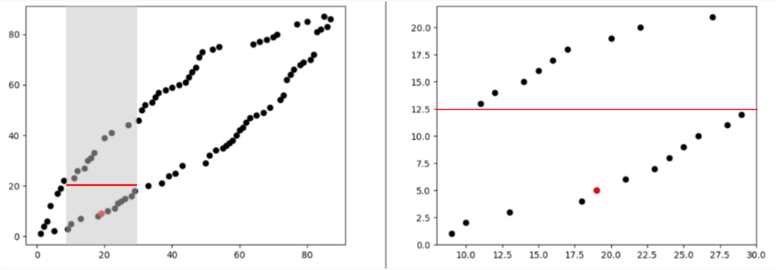
- We adapt a local limit result for Galton-Watson trees to know the positions of the leaves.



THE CONSTRUCTION OF THE RANDOM ORDER $\boldsymbol{\sigma}_{321}^{\infty}$.

- We consider the classical total order on $\mathbb{Z}$;

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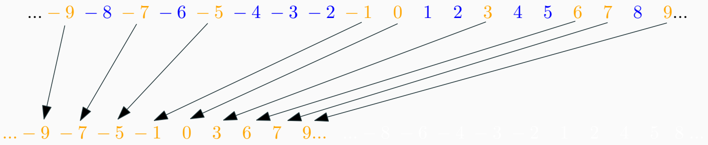

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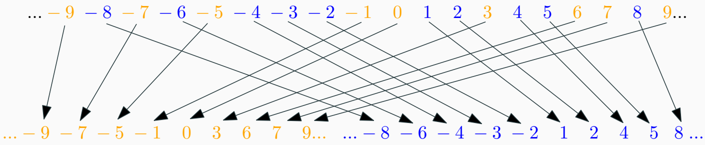

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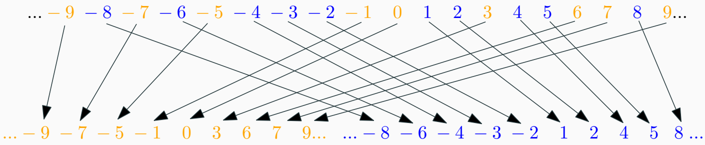
- The new random order has the same distribution as $\boldsymbol{\sigma}^{\infty}_{321}$.



Our goal

Study limits of random permutations when the size tends to infinity

1. Scaling limits:
   Limiting objects: Permutons
   Corresponding statistic: $\widetilde{\mathrm{occ}}(\pi, \sigma)$, for all $\pi \in \mathcal{S}$
   Concrete examples: ρ-avoiding permutations for $|\pi| = 3$, separable permutations, substitution-closed classes, Mallows permutations, ...
2. Local limits:
   Limiting objects: Rooted permutations + shift-invariant property
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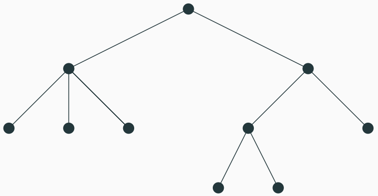
   Future projects: Substitution-closed permutation classes(?), Mallows permutations(?), ...

Rooted ordered tree

The trees that we consider are rooted and ordered.



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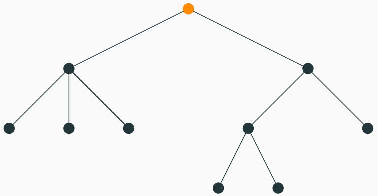

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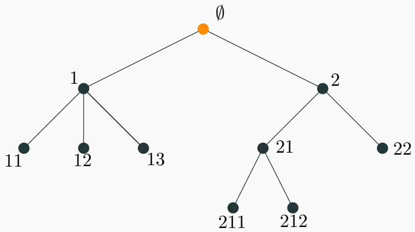

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Recall that the Galton–Watson tree is called subcritical, critical or supercritical when the expected number of children $\mathbb{E}[\xi] < 1$, $\mathbb{E}[\xi] = 1$ or $\mathbb{E}[\xi] > 1$. It is a standard basic fact of branching process theory that T is *a.s.* finite if $\mathbb{E}[\xi] \leq 1$, but T is infinite with positive probability if $\mathbb{E}[\xi] > 1$ (the supercritical case).

Local limit for uniform ρ-avoiding permutations with $|\rho| = 3$

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We say that a permutation σ avoids a pattern $\rho \in \mathcal{S}$ if

$$\mathrm{occ}(\rho, \sigma) = 0.$$

We underline that the definition of ρ-avoiding permutation refers to patterns and not to consecutive patterns.

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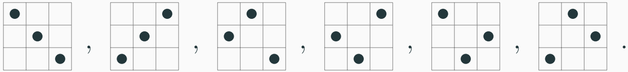
$\rho =$ 

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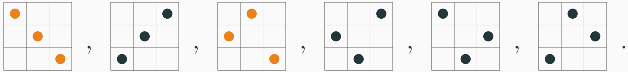
$\rho =$ 

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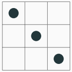
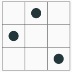
$\rho =$  ,  .

The *shift invariant* property

Definition

A random infinite rooted permutation $(\mathbb{Z}, \preccurlyeq)$ has the shift invariant property if for all patterns $\pi \in \mathcal{S}$,

$$\mathbb{P}(\pi_1 \preccurlyeq \pi_2 \preccurlyeq \ldots \preccurlyeq \pi_k) = \mathbb{P}(\pi_1 + s \preccurlyeq \pi_2 + s \preccurlyeq \ldots \preccurlyeq \pi_k + s), \quad \forall s \in \mathbb{Z}.$$

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$$\mathbb{P}(\pi_1 \preccurlyeq \pi_2 \preccurlyeq \ldots \preccurlyeq \pi_k) = \mathbb{P}(\pi_1 + s \preccurlyeq \pi_2 + s \preccurlyeq \ldots \preccurlyeq \pi_k + s), \quad \forall s \in \mathbb{Z}.$$

Example

Let $(\mathbb{Z}, \preccurlyeq)$ be a random shift-invariant rooted permutation. If $\pi = 132$ then

$$\mathbb{P}(1 \preccurlyeq 3 \preccurlyeq 2)$$

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Proposition

Let $(\mathbb{Z}, \preccurlyeq)$ be the annealed Benjamini-Schramm limit of a sequence $(\boldsymbol{\sigma}^n)_{n\in\mathbb{N}}$ of random permutations, then $(\mathbb{Z}, \preccurlyeq)$ has the shift invariant property.

The *shift invariant* property

QUESTION: Is every shift invariant random infinite rooted permutation $(\mathbb{Z}, \preccurlyeq)$ the annealed Benjamini-Schramm limit of some sequence of random permutations?

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Theorem [B.]

Let $(\mathbb{Z}, \preccurlyeq)$ be a random shift-invariant rooted permutation. Then the sequence of random permutations $(\boldsymbol{\sigma}^n)_{n\in\mathbb{N}}$ defined, for all $n \in \mathbb{N}$, by

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Curiosity

The corresponding property for graphs is called unimodularity and the following problem still open:
Is every unimodular random graph the local limit in distribution of uniformly pointed random graphs? (solved for trees).

Basics on Permutations

Permutations

Permutation of size $n \equiv$ Bijection from $[n] = \{1, \ldots, n\}$ to itself.
Set $\mathcal{S}^n$ and $\mathcal{S} = \cup_{n \in \mathbb{N}} \mathcal{S}^n$.

NOTATION:

- Two lines:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 2 & 4 & 8 & 1 & 6 & 3 & 7 \end{pmatrix}$$

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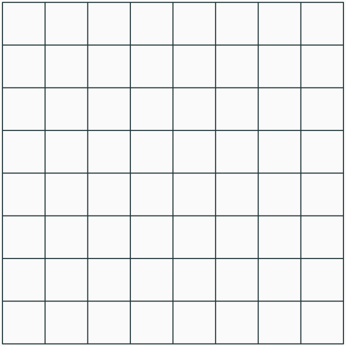

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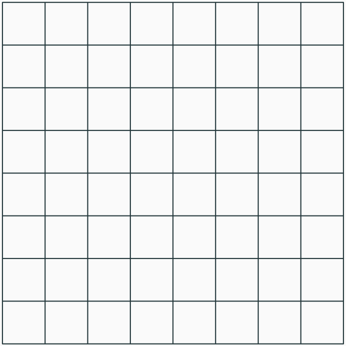

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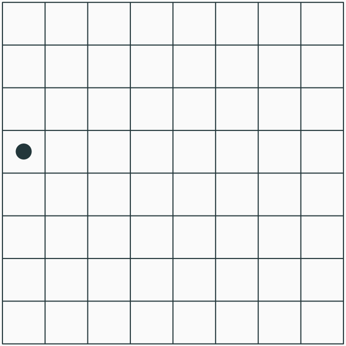

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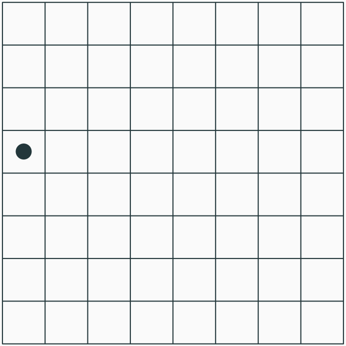

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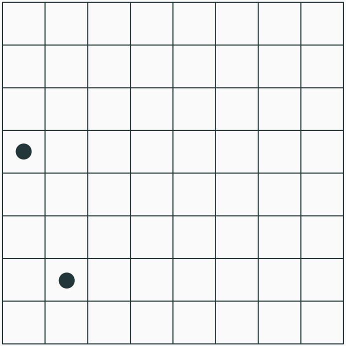

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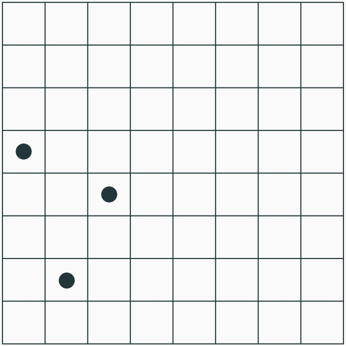

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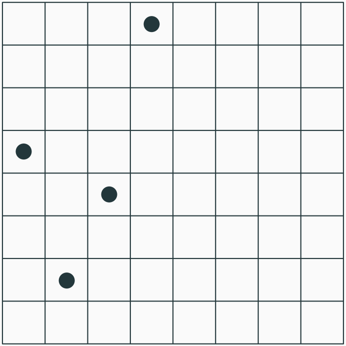

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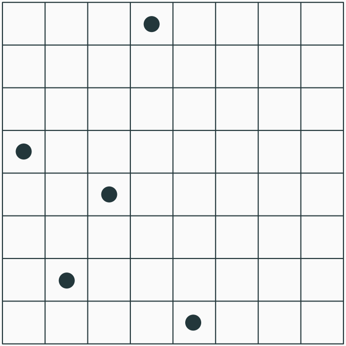

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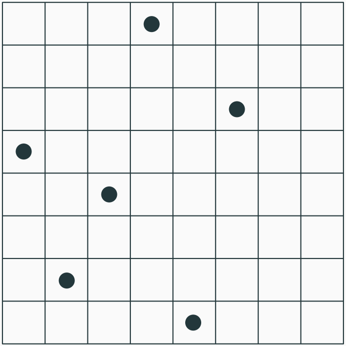

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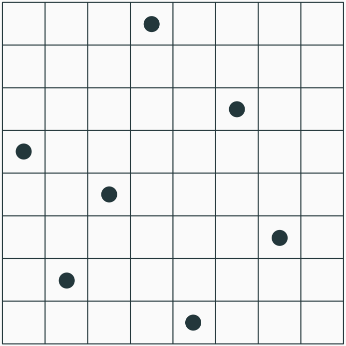

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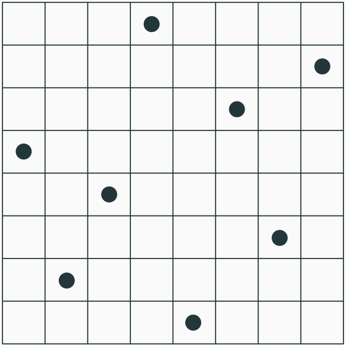

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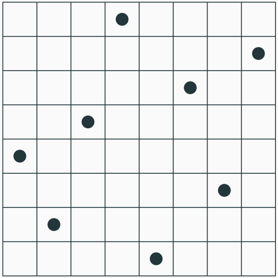

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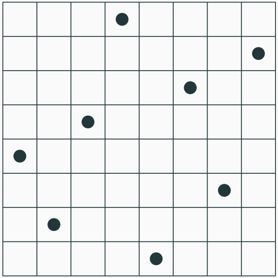

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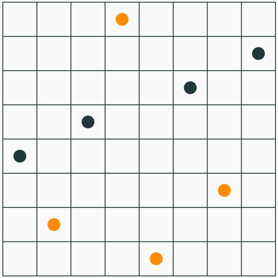

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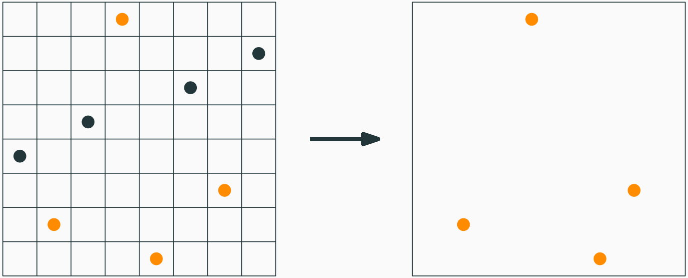

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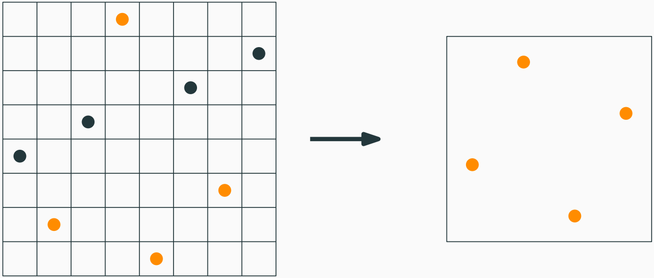

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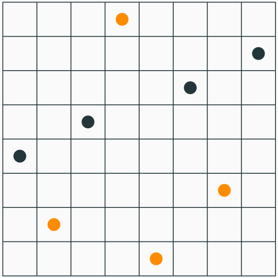

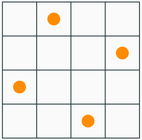

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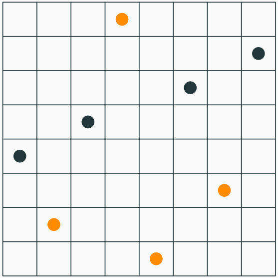

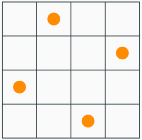

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π is a pattern of σ if there exists I such that $\mathrm{pat}_I(\sigma) = \pi$.

Permutation patterns

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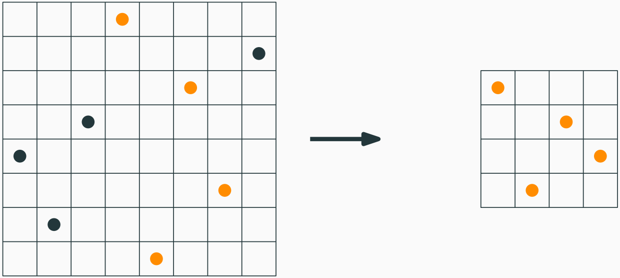

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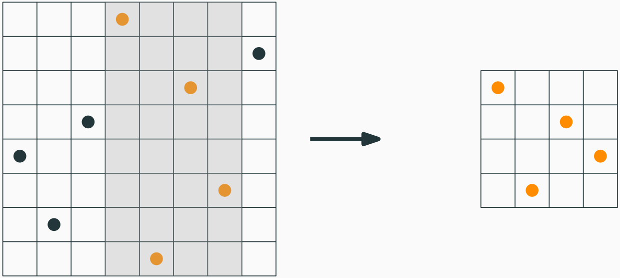

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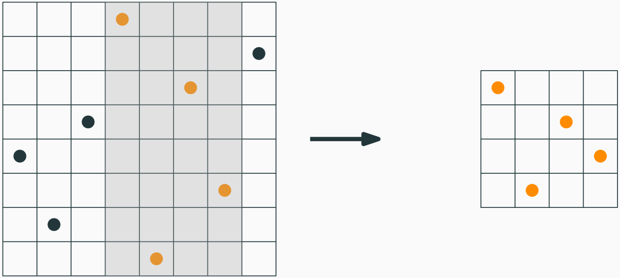

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Definition

π is a pattern of σ if there exists I such that $\mathrm{pat}_I(\sigma) = \pi$. Moreover, if I is an interval then π is a consecutive pattern in σ.

Definition

We denote by $\text{occ}(\pi, \sigma)$ the number of occurrences of a pattern π in σ. More formally, if $\pi \in \mathcal{S}^k$ and $\sigma \in \mathcal{S}^n$,

$$\text{occ}(\pi, \sigma) = \text{Card}\{I \subset [n] \text{ of cardinality } k \text{ such that } \text{pat}_I(\sigma) = \pi\}.$$

Moreover we denote by $\widetilde{\text{occ}}(\pi, \sigma)$ the proportion of occurrences of a pattern π in σ namely

$$\widetilde{\text{occ}}(\pi, \sigma) = \frac{\text{occ}(\pi, \sigma)}{\binom{n}{k}}.$$

Definition

We denote by $c\text{-}occ(\pi, \sigma)$ the number of consecutive occurrences of a pattern π in σ. More formally, if $\pi \in \mathcal{S}^k$ and $\sigma \in \mathcal{S}^n$,

$$c\text{-}occ(\pi, \sigma) = \text{Card}\left\{ I \subset [n] \middle| I \text{ is an } \underline{\text{interval}}, \text{Card}(I) = k, \text{pat}_I(\sigma) = \pi \right\}.$$

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Pattern densities

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The natural choice should be $n-k+1$

Weak-local convergence: the random case

$$\sigma^n \xrightarrow{BS} \boldsymbol{\sigma}^\infty \overset{def}{\iff} (\sigma^n, \boldsymbol{i}_n) \xrightarrow{law} \boldsymbol{\sigma}^\infty .$$

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QUESTION: What happens if the sequence $(\boldsymbol{\sigma}^n)_{n\in\mathbb{N}}$ is random?

$$\sigma^n \xrightarrow{BS} \boldsymbol{\sigma}^\infty \overset{def}{\Longleftrightarrow} (\sigma^n, \boldsymbol{i}_n) \xrightarrow{law} \boldsymbol{\sigma}^\infty.$$

Definition

We say that a sequence $(\boldsymbol{\sigma}^n)_{n\in\mathbb{N}}$ of random permutations converges in the annealed Benjamini-Schramm sense to a random rooted permutation $\boldsymbol{\sigma}^\infty$ if

$$(\boldsymbol{\sigma}^n, \boldsymbol{i}_n)_{n\in\mathbb{N}} \xrightarrow{law} \boldsymbol{\sigma}^\infty, \quad \text{w.r.t. the local distance d.}$$

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We say that a sequence $(\boldsymbol{\sigma}^n)_{n\in\mathbb{N}}$ of random permutation converges in the quenched Benjamini-Schramm sense to a <u>random measure</u> $\boldsymbol{\mu}^\infty$ on $\tilde{\mathcal{S}}_\bullet$ if

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We write $\boldsymbol{\sigma}^n \xrightarrow{aBS} \boldsymbol{\sigma}^\infty$ and $\boldsymbol{\sigma}^n \xrightarrow{qBS} \boldsymbol{\mu}^\infty$.