Rook and Wilf equivalence of integer partitions

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(Joint work with Dan Saracino & Nathan McNew)

Permutation Patterns – Dartmouth College
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A **partition of** n is a weakly decreasing sequence λ

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_h > 0$$

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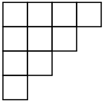
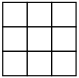
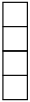
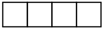
★ We identify partitions with **Ferrers diagrams**. E.g.,

$4+3+2+1 =$ 

$3+3+3 =$ 

$1+1+1+1 =$ 

$4 =$ 

Rook Theory

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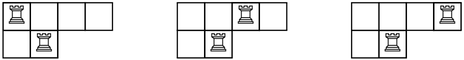
Question: How many configurations of k non-attacking rooks can be placed on a partition (Ferrers diagram)?



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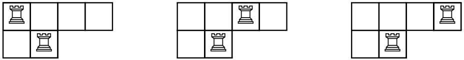

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For any partition $\mu \in \mathbb{P}$ we define its **rook polynomial** to be

$$R_\mu(q) = \sum_{k \geq 0} r(\mu, k) q^k$$

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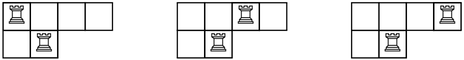

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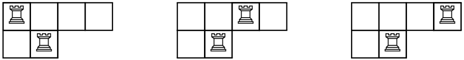

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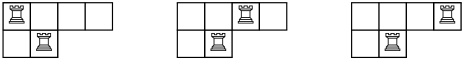

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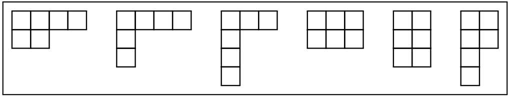

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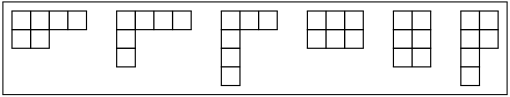

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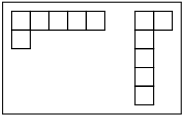

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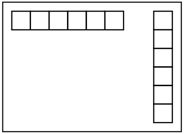
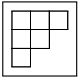


$1 + 6q + 7q^2 + q^3$



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$$\{1+\mu_1, 2+\mu_2, 3+\mu_3, \ldots\} = \{1+\tau_1, 2+\tau_2, 3+\tau_3, \ldots\}$$

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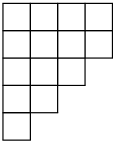

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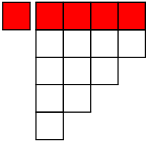

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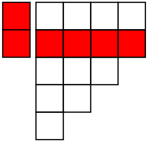

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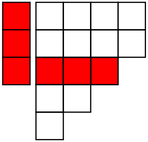

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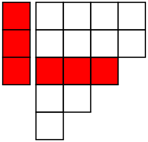

- Hence the term *L*-multisets

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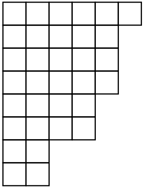
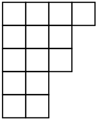
$\alpha =$  **contains** $\mu =$ 

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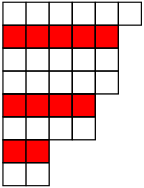
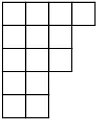
$\alpha =$  **contains** $\mu =$ 

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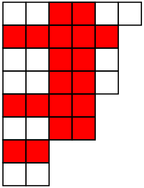
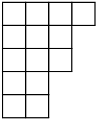
$\alpha =$  **contains** $\mu =$ 

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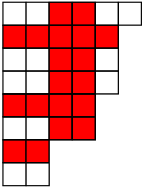
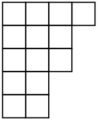
$\alpha =$  **contains** $\mu =$ 

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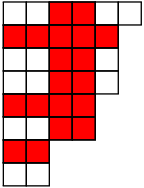
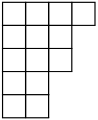
$\alpha =$  **contains** $\mu =$ 

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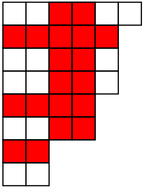
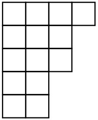
$\alpha =$  **contains** $\mu =$ 

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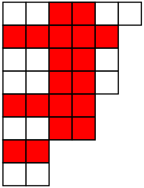
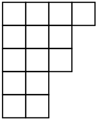
$\alpha =$  **contains** $\mu =$ 

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(Containment defined by row-only deletion?
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- $\mathbb{P}_n(\mu, k) = \{\alpha \in \mathbb{P}_n(\mu) \mid \alpha_1 = \mu_1 + k\}$

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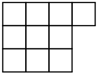
If $\mu =$  $\in \mathbb{P}_{10}$, then we have the following sets:

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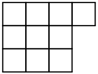
If $\mu =$  $\in \mathbb{P}_{10}$, then we have the following sets:

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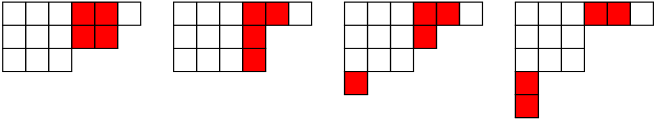

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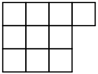
If $\mu =$  $\in \mathbb{P}_{10}$, then we have the following sets:

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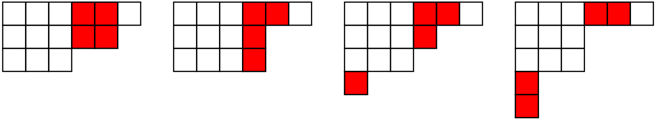
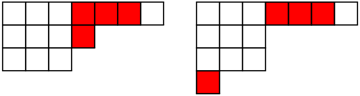


and

$\mathbb{P}_{14}(\mu, 3)$



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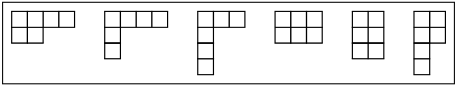
Refining this, μ, τ are **width-Wilf equivalent** provided

$$P_{\mu,k}(q) = P_{\tau,k}(q) \qquad (\text{for all } k \geq 0)$$

Integer partition patterns

Wilf classes for $n = 6$:



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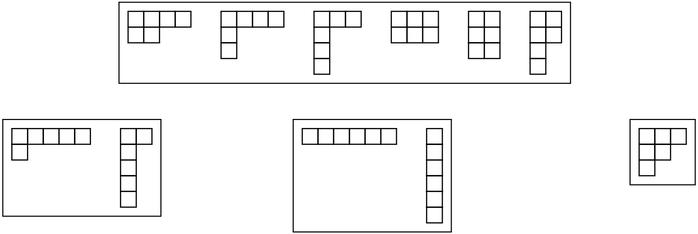

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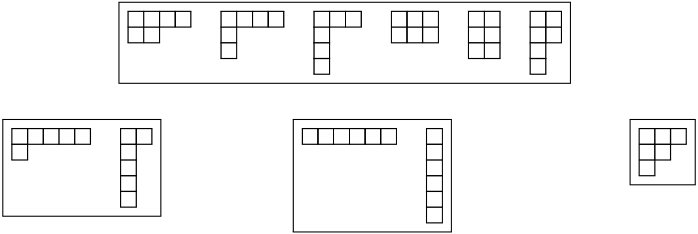

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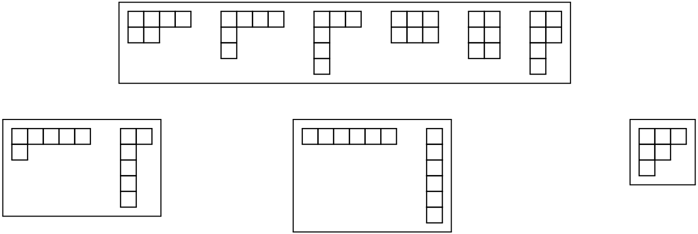

★ Same as rook classes! Can we prove it?

Our main result

Theorem (Bloom, Saracino)

$$\textit{Rook equivalence} \iff \textit{Wilf equivalence.}$$

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Theorem (Bloom, Saracino)

Rook equivalence $\iff$ *Wilf equivalence.*

$$\left(\begin{array}{l} \textit{Also, assuming } \mu_1 = \tau_1 \textit{ for } \mu, \tau \in \mathbb{P}, \textit{ then} \\ \qquad \textit{rook equivalence} \iff \textit{width-Wilf equivalence} \\ \qquad P_{\mu,1}(q) = P_{\tau,1}(q) \iff \textit{width-Wilf equivalent} \end{array}\right)$$

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Theorem (Bloom, Saracino)

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($\Longrightarrow$) Characterize $P_{\mu,k}(q)$ in terms of "L-multisets".

Proof sketch: rook $\implies$ Wilf

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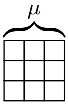

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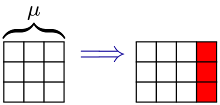

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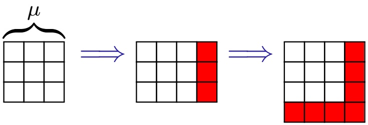

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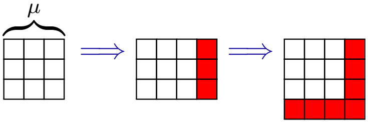

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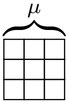

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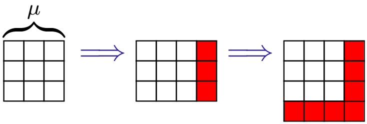

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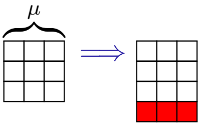

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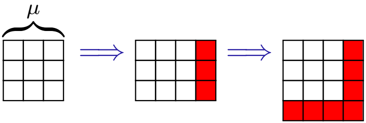

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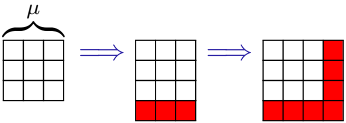

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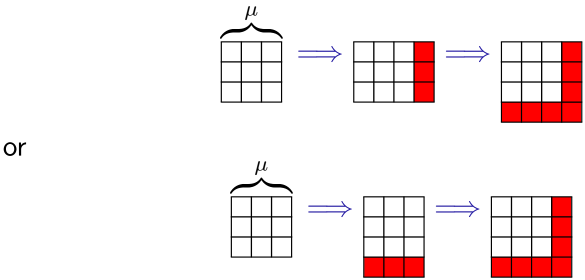

★ Need Inclusion/Exclusion!

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$$(2,2,2,1) + (3,2,2,1) = (5,4,4,2)$$

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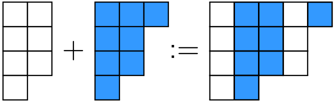

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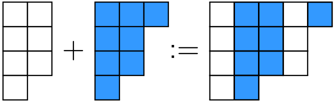

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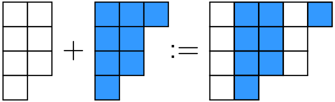

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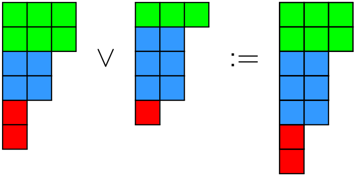

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Proof sketch: rook $\implies$ Wilf

Definition

For any (finite) $S \subset \mathbb{P}$ we define $\vee_S : \mathbb{P} \to \mathbb{P}$ as

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  $\Rightarrow$ Characterize when the operators

$$\vee_S = \vee_T$$

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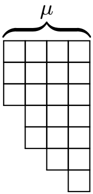

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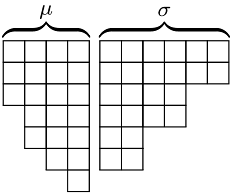

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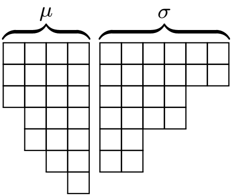

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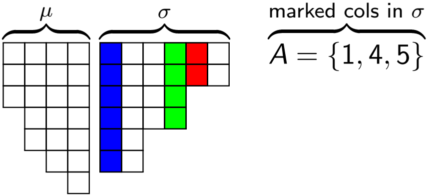

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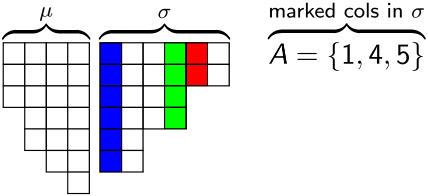

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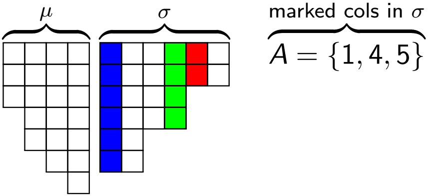

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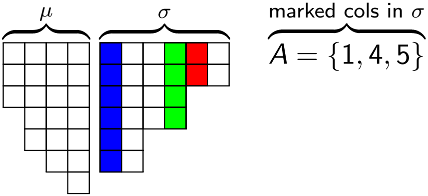

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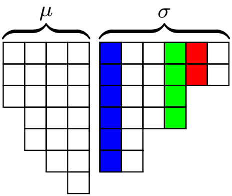

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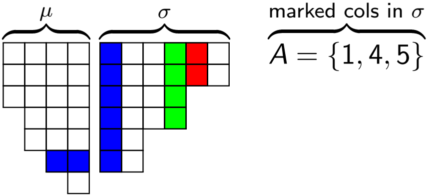

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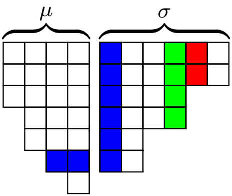

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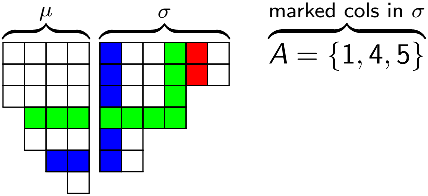

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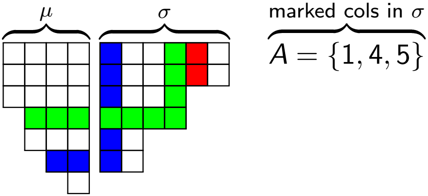

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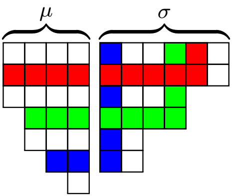

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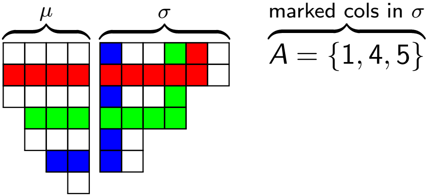

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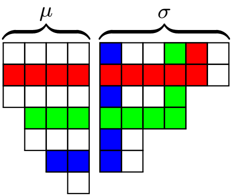

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★ Marked structures "index" L-multisets!

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after loads of cancellation...

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Let $\mu \in \mathbb{P}$ *so that* $|\mu_i - \mu_j| > 1$ *for* $i \neq j$*, then*

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Fix $N \geq 0$. Then the GF for partitions τ with $|\tau_i - \tau_k| < N$ is rational.

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$\Longrightarrow \tau$ *avoids* $\mu = (N+1, 1)$

Thank You!