Patterns in Standard Young Tableaux

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Based on joint work with:
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Outline

Background on Standard Young Tableaux

q-enumeration of SYT's via major index

Distribution Question: From Combinatorics to Probability

Existence Question: New Posets on Tableaux

Unimodality Question: ???

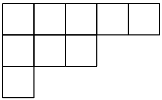
Partitions

Def. A *partition* of a number n is a weakly decreasing sequence of positive integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$$

such that $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k = |\lambda|$. Write $\lambda \vdash n$.

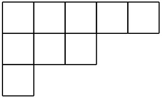
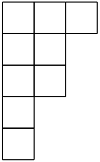
Partitions can be visualized by their *Ferrers diagram*

$(5, 3, 1) \longrightarrow$ 

The *cells* are indexed by matrix coordinates (i, j) so $(1, 5)$ is the rightmost cell in the top row.

Conjugate Partition

Def. The *conjugate* of a partition $\lambda \vdash n$ is the partition $\lambda' \vdash n$ whose parts count the number of cells in each column of λ.

$\lambda = (5,3,1) =$  and $\lambda' = (3,2,2,1,1) =$ 

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Filling Partitions

Defn. A map from the cells of λ to the positive integers is a *filling* of λ.

1	3	7	2	8
6	1	2		
5				

Defn. A filling of $\lambda \vdash n$ is *bijective* if every number in $[n] = \{1, 2, \ldots, n\}$ appears exactly once.

1	3	7	4	9
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Question. How many bijective fillings are there of shape $(5, 3, 1)$?

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Question. How many bijective fillings are there of shape $(5, 3, 1)$?

Answer. $9! = 362,880$. Bijection with permutations of 9.

Standard Young Tableaux

Defn. A *standard Young tableaux* of shape λ is a bijective filling of λ such that every row is increasing from left to right and every column is increasing from top to bottom.

1	3	6	7	9
2	5	8		
4				

Important Fact. The standard Young tableaux of shape λ, denoted $\mathrm{SYT}(\lambda)$, index a basis of the irreducible S_n representation indexed by λ.

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Important Fact. The standard Young tableaux of shape λ, denoted $\mathrm{SYT}(\lambda)$, index a basis of the irreducible S_n representation indexed by λ.

Question. How many standard Young tableaux are there of shape $(5,3,1)$? **Answer.** $\#\mathrm{SYT}(5,3,1) = 162$

Standard Young Tableaux

Pause: Find all standard Young tableaux on $(2, 2)$.

Counting Standard Young Tableaux

Hook Length Formula. (Frame-Robinson-Thrall, 1954)
If λ is a partition of n, then

$$\#SYT(\lambda) = \frac{n!}{\prod_{c \in \lambda} h_c}$$

where h_c is the *hook length* of the cell c, i.e. the number of cells directly to the right of c or below c, including c.

Example. Filling cells of $\lambda = (5,3,1) \vdash 9$ by hook lengths:

7	5	4	2	1
4	2	1		
1				

So, $\#SYT(5,3,1) = \frac{9!}{7\cdot5\cdot4\cdot2\cdot4\cdot2} = 162.$

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Remark. Notable other proofs by Greene-Nijenhuis-Wilf '79 (probabilistic), Eriksson '93 (bijective), Krattenthaler '95 (bijective), Novelli -Pak -Stoyanovskii'97 (bijective), Bandlow'08,

q-Counting Standard Young Tableaux

Def. The *descent set* of a standard Young tableaux T, denoted $D(T)$, is the set of positive integers i such that $i+1$ lies in a row strictly below the cell containing i in T.

The *major index* of T is the sum of its descents:

$$\mathrm{maj}(T) = \sum_{i \in D(T)} i.$$

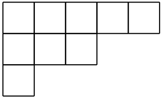
Example. The descent set of T is $D(T) = \{1, 3, 4, 7\}$ so $\mathrm{maj}(T) = 15$ for $T = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 6 & 7 & 9 \\ \hline 2 & 4 & 8 \\ \cline{1-3} 5 \\ \cline{1-1}\end{array}$.

Def. The *major index generating function* for λ is

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) := \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)}$$

q-Counting Standard Young Tableaux

Example. $\lambda = (5,3,1)$



$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) := \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} =$

$$q^{23} + 2q^{22} + 4q^{21} + 5q^{20} + 8q^{19} + 10q^{18} + 13q^{17} + 14q^{16} + 16q^{15}$$

$$+16q^{14} + 16q^{13} + 14q^{12} + 13q^{11} + 10q^{10} + 8q^{9} + 5q^{8} + 4q^{7} + 2q^{6} + q^{5}$$

Note, at $q = 1$, we get back 162.

q-Counting Standard Young Tableaux

Thm.(Lusztig-Stanley 1979) Given a partition $\lambda \vdash n$, say

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) := \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = \sum_{k \geq 0} b_{\lambda,k} q^k.$$

Then $b_{\lambda,k} := \#\{T \in \mathrm{SYT}(\lambda) : \mathrm{maj}(T) = k\}$ is the number of times the irreducible S_n module indexed by λ appears in the decomposition of the coinvariant algebra $\mathbb{Z}[x_1, x_2, \ldots, x_n]/I_+$ in the homogeneous component of degree k.

Comments.

- The "*fake degree sequence*" is $(b_{\lambda,0}, b_{\lambda,1}, b_{\lambda,2}, \ldots)$.

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Comments.

- The "*fake degree sequence*" is $(b_{\lambda,0}, b_{\lambda,1}, b_{\lambda,2}, \ldots)$.
- The fake degrees also appear in branching rules between symmetric groups and cyclic subgroups (Stembridge, 1989), and the degree polynomials of certain irreducible $\mathrm{GL}_n(\mathbb{F}_q)$-representations (Steinberg 1951, Green 1955).

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Def. The *descent set* of a standard Young tableaux T, denoted $D(T)$, is the set of positive integers i such that $i+1$ lies in a row strictly below the cell containing i in T.

The *major index* of T is the sum of its descents:

$$\mathrm{maj}(T) = \sum_{i \in D(T)} i.$$

Example. There are 2 standard Young tableaux of shape $(2,2)$:

$$S = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \qquad T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

$D(S) = \{2\}$ and $D(T) = \{1,3\}$ so $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = q^2 + q^4$.
Represent $q^2 + q^4$ by the vector of coefficients (00101).

q-Counting Standard Young Tableaux

Examples. $(2,2) \vdash 4$: $(0\ 0\ 1\ 0\ 1)$

$(5,3,1)$: $(00000\ 1\ 2\ 4\ 5\ 8\ 10\ 13\ 14\ 16\ 16\ 16\ 14\ 13\ 10\ 8\ 5\ 4\ 2\ 1)$

q-Counting Standard Young Tableaux

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$(6,4) \vdash 10$: $(0\ 0\ 0\ 0\ 1\ 1\ 2\ 2\ 4\ 4\ 6\ 6\ 8\ 7\ 8\ 7\ 8\ 6\ 6\ 4\ 4\ 2\ 2\ 1\ 1)$

$(6,6) \vdash 12$: $(0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 2\ 2\ 4\ 3\ 5\ 5\ 7\ 6\ 9\ 7\ 9\ 8\ 9\ 7\ 9\ 6\ 7\ 5$
$5\ 3\ 4\ 2\ 2\ 1\ 1\ 0\ 1)$

$(11,5,3,1) \vdash 20$: (1 3 8 16 32 57 99 160 254 386 576 832 1184
1645 2255 3031 4027 5265 6811 8689 10979 13706 16959 20758
25200 30296 36143 42734 50163 58399 67523 77470 88305 99925
112370 125492 139307 153624 168431 183493 198778 214017
229161 243913 258222 271780 284542 296200 306733 315853
323571 329629 334085 336727 337662 336727 334085 329629
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Key Questions for $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$

Recall $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \sum b_{\lambda,k} q^k$.

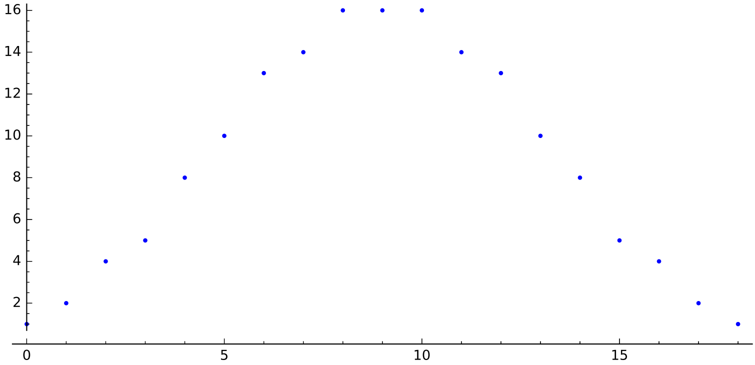
Distribution Question. What patterns do the coefficients in the list $(b_{\lambda,0}, b_{\lambda,1}, \ldots)$ exhibit?

Existence Question. For which λ, k does $b_{\lambda,k} = 0$?

Unimodality Question. For which λ, are the coefficients of $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ *unimodal*, meaning

$$b_{\lambda,0} \leq b_{\lambda,1} \leq \ldots \leq b_{\lambda,m} \geq b_{\lambda,m+1} \geq \ldots ?$$

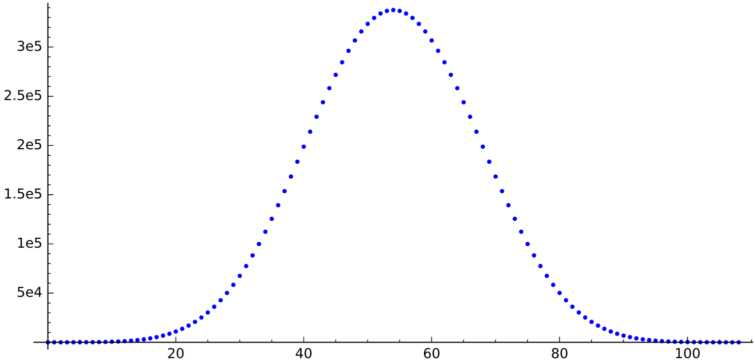
Visualizing Major Index Generating Functions



Visualizing the coefficients of $\mathrm{SYT}(5,3,1)^{\mathrm{maj}}(q)$:

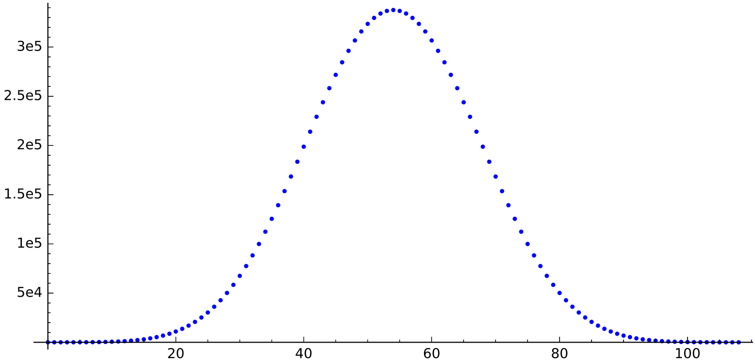
$$(1,2,4,5,8,10,13,14,16,16,16,14,13,10,8,5,4,2,1)$$

Visualizing Major Index Generating Functions



Visualizing the coefficients of $\mathrm{SYT}(11,5,3,1)^{\mathrm{maj}}(q)$.

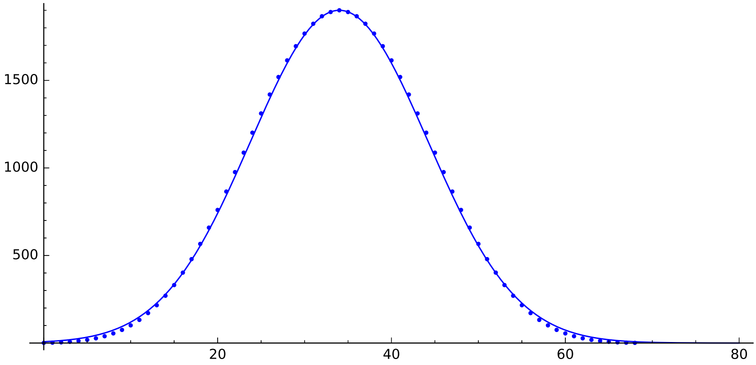
Visualizing Major Index Generating Functions



Visualizing the coefficients of $\mathrm{SYT}(11,5,3,1)^{\mathrm{maj}}(q)$.

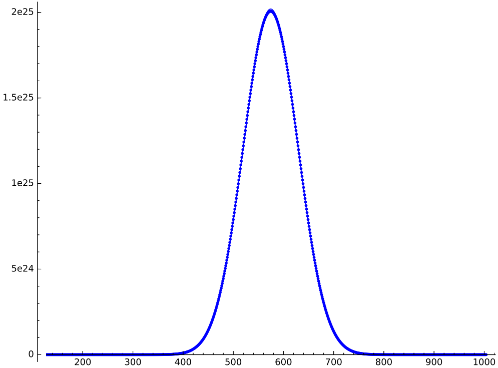
Question. What type of curve is that?

Visualizing Major Index Generating Functions



Visualizing the coefficients of $\mathrm{SYT}(10,6,1)^{\mathrm{maj}}(q)$ along with the Normal distribution with $\mu = 34$ and $\sigma^2 = 98$.

Visualizing Major Index Generating Functions



Visualizing the coefficients of $\mathrm{SYT}(8,8,7,6,5,5,5,2,2)^{\mathrm{maj}}(q)$

"Fast" Computation of $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$

Thm.(Stanley's q-analog of the Hook Length Formula for $\lambda \vdash n$)

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \frac{q^{b(\lambda)}[n]_q!}{\prod_{c\in\lambda}[h_c]_q}$$

where

- $b(\lambda) := \sum(i-1)\lambda_i$
- h_c is the hook length of the cell c
- $[n]_q := 1 + q + \cdots + q^{n-1} = \frac{q^n-1}{q-1}$
- $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$

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The Trick. Each q-integer $[n]_q$ factors into a product of *cyclotomic polynomials* $\Phi_d(q)$,

$$[n]_q = 1 + q + \cdots + q^{n-1} = \prod_{d|n}\Phi_d(q).$$

Cancel all of the factors from the denominator of $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ from the numerator, and then expand the remaining product.

Corollaries of Stanley's formula

Thm.(Stanley's q-analog of the Hook Length Formula for $\lambda \vdash n$)

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \frac{q^{b(\lambda)}[n]_q!}{\prod_{c \in \lambda}[h_c]_q}$$

Corollaries.

1. $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \mathrm{SYT}(\lambda')^{\mathrm{maj}}(q)$.
2. The coefficients of $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ are symmetric.
3. There is a unique min-maj and max-maj tableau of shape λ.

Min-Maj and Max-Maj Tableaux

Example. The *min-maj* and *max-maj* tableaux for $(6, 4, 3, 3, 1)$.

1	3	4	11	$\boxed{16}$	$\boxed{17}$
2	6	7	$\boxed{15}$		
5	9	10			
8	$\boxed{13}$	$\boxed{14}$			
$\boxed{12}$					

$$b(\lambda) = \sum (i-1)\lambda_i = 23$$

1	2	3	5	9	$\boxed{13}$
4	6	10	$\boxed{14}$		
7	11	$\boxed{15}$			
8	12	$\boxed{16}$			
$\boxed{17}$					

$$\binom{17}{2} - b(\lambda') = 109$$

Converting q-Enumeration to Discrete Probability

Vic Reiner's Quote:

> *"If we can count it, we should also try to q-count it."*

I say:

> *"If we can q-count it, we should try to probabalize it."*

Converting q-Enumeration to Discrete Probability

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"If we can count it, we should also try to q-count it."

I say:

"If we can q-count it, we should try to probabalize it."

If $f(q) = a_0 + a_1 q + a_2 q^2 + \cdots + a_n q^n$ where a_i are nonnegative integers, then construct the random variable X_f with discrete probability distribution

$$\mathbb{P}(X_f = k) = \frac{a_k}{\sum_j a_j} = \frac{a_k}{f(1)}.$$

Now, if f is part of a family of q-analogs, we can study the limiting distributions.

Converting q-Enumeration to Discrete Probability

Example. For $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \sum b_{\lambda,k} q^k$, define the integer random variable $X_\lambda[\mathrm{maj}]$ with discrete probability distribution

$$\mathbb{P}(X_\lambda[\mathrm{maj}] = k) = \frac{b_{\lambda,k}}{|\mathrm{SYT}(\lambda)|}.$$

We claim the distribution of $X_\lambda[\mathrm{maj}]$ "usually" is approximately normal for most shapes λ. Let's make that precise!

Standardization

Thm.(Adin-Roichman, 2001)
For any partition λ, the mean and variance of $X_\lambda[\mathrm{maj}]$ are

$$\mu_\lambda = \frac{\binom{|\lambda|}{2} - b(\lambda') + b(\lambda)}{2} = b(\lambda) + \frac{1}{2}\left[\sum_{j=1}^{|\lambda|} j - \sum_{c\in\lambda} h_c\right],$$

and

$$\sigma_\lambda^2 = \frac{1}{12}\left[\sum_{j=1}^{|\lambda|} j^2 - \sum_{c\in\lambda} h_c^2\right].$$

Def. The *standardization* of $X_\lambda[\mathrm{maj}]$ is

$$X_\lambda^*[\mathrm{maj}] = \frac{X_\lambda[\mathrm{maj}] - \mu_\lambda}{\sigma_\lambda}.$$

So $X_\lambda^*[\mathrm{maj}]$ has mean 0 and variance 1 for any λ.

Asymptotic Normality

Def. Let $X_1, X_2, \ldots$ be a sequence of real-valued random variables with standardized cumulative distribution functions $F_1(t), F_2(t), \ldots$. The sequence is *asymptotically normal* if

$$\forall t \in \mathbb{R}, \quad \lim_{n \to \infty} F_n(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2} = \mathbb{P}(N < t)$$

where N is a Normal random variable with mean 0 and variance 1.

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where N is a Normal random variable with mean 0 and variance 1.

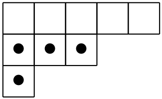
Question. In what way can a sequence of partitions approach infinity?

The Aft Statistic

Def. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, let

$$\mathrm{aft}(\lambda) := n - \max\{\lambda_1, k\}.$$

Example. $\lambda = (5, 3, 1)$ then $\mathrm{aft}(\lambda) = 4$.



Look it up: Aft is now on FindStat as St001214

Distribution Question: From Combinatorics to Probability

Thm. (Billey-Konvalinka-Swanson, 2018+)

Suppose $\lambda^{(1)}, \lambda^{(2)}, \ldots$ is a sequence of partitions, and let $X_N := X_{\lambda^{(N)}}[\mathrm{maj}]$ be the corresponding random variables for the maj statistic. Then, the sequence $X_1, X_2, \ldots$ is asymptotically normal if and only if $\mathrm{aft}(\lambda^{(N)}) \to \infty$ as $N \to \infty$.

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Question. What happens if $\mathrm{aft}(\lambda^{(N)})$ does not go to infinity as $N \to \infty$?

Distribution Question: From Combinatorics to Probability

Thm.(Billey-Konvalinka-Swanson, 2018+)
Let $\lambda^{(1)}, \lambda^{(2)}, \ldots$ be a sequence of partitions. Then $(X_{\lambda^{(N)}}[\mathrm{maj}]^*)$ converges in distribution if and only if

(i) $\mathrm{aft}(\lambda^{(N)}) \to \infty$; or

(ii) $|\lambda^{(N)}| \to \infty$ and $\mathrm{aft}(\lambda^{(N)})$ is eventually constant; or

(iii) the distribution of $X^*_{\lambda^{(N)}}[\mathrm{maj}]$ is eventually constant.

The limit law is $\mathcal{N}(0,1)$ in case (i), Σ^*_M in case (ii), and discrete in case (iii).

Here Σ_M denotes the sum of M independent identically distributed uniform $[0,1]$ random variables, known as the Irwin–Hall distribution or the *uniform sum distribution*.

Proof ideas: Characterize the Moments and Cumulants

Definitions.

- For $d \in \mathbb{Z}_{\geq 0}$, the *dth moment*

$$\mu_d := \mathbb{E}[X^d]$$

- The *moment-generating function* of X is

$$M_X(t) := \mathbb{E}[e^{tX}] = \sum_{d=0}^{\infty} \mu_d \frac{t^d}{d!},$$

- The *cumulants* $\kappa_1, \kappa_2, \ldots$ of X are defined to be the coefficients of the exponential generating function

$$K_X(t) := \sum_{d=1}^{\infty} \kappa_d \frac{t^d}{d!} := \log M_X(t) = \log \mathbb{E}[e^{tX}].$$

Nice Properties of Cumulants

1. *(Familiar Values)* The first two cumulants are $\kappa_1 = \mu$, and $\kappa_2 = \sigma^2$.
2. *(Shift Invariance)* The second and higher cumulants of X agree with those for $X - c$ for any $c \in \mathbb{R}$.
3. *(Homogeneity)* The dth cumulant of cX is $c^d \kappa_d$ for $c \in \mathbb{R}$.
4. *(Additivity)* The cumulants of the sum of *independent* random variables are the sums of the cumulants.
5. *(Polynomial Equivalence)* The cumulants and moments are determined by polynomials in the other sequence.

Examples of Cumulants and Moments

Example. Let $X = \mathcal{N}(\mu, \sigma^2)$ be the normal random variable with mean μ and variance σ^2. Then the cumulants are

$$\kappa_d = \begin{cases} \mu & d = 1, \\ \sigma^2 & d = 2, \\ 0 & d \geq 3. \end{cases}$$

and for $d > 1$,

$$\mu_d = \begin{cases} 0 & \text{if } d \text{ is odd}, \\ \sigma^d (d-1)!! & \text{if } d \text{ is even}. \end{cases}$$

.

Example. For a Poisson random variable X with mean μ, the cumulants are all $\kappa_d = \mu$, while the moments are $\mu_d = \sum_{i=1}^{d} \mu^i S_{i,d}$.

Cumulants for Major Index Generating Functions

Thm.(Billey-Konvalinka-Swanson, 2018+)
Let $\lambda \vdash n$ and $d \in \mathbb{Z}_{>1}$. We have

$$\kappa_d^\lambda = \frac{B_d}{d}\left[\sum_{j=1}^{n} j^d - \sum_{c\in\lambda} h_c^d\right] \quad (1)$$

where $B_0, B_1, B_2, \ldots = 1, \frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, \ldots$ are the Bernoulli numbers (OEIS A164555 / OEIS A027642).

Remark. We use this theorem to prove that as aft approaches infinity the standardized cumulants for $d \geq 3$ all go to 0 proving the Asymptotic Normality Theorem.

Cumulants for Major Index Generating Functions

Thm.(Billey-Konvalinka-Swanson, 2018+)
Let $\lambda \vdash n$ and $d \in \mathbb{Z}_{>1}$. We have

$$\kappa_d^\lambda = \frac{B_d}{d}\left[\sum_{j=1}^{n} j^d - \sum_{c\in\lambda} h_c^d\right] \qquad (1)$$

where $B_0, B_1, B_2, \ldots = 1, \frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, \ldots$ are the Bernoulli numbers (OEIS A164555 / OEIS A027642).

Remark. We use this theorem to prove that as aft approaches infinity the standardized cumulants for $d \geq 3$ all go to 0 proving the Asymptotic Normality Theorem.

Remark. Note, κ_2^λ is exactly the Adin-Roichman variance formula.

q-Enumeration to Probability

Thm.(Chen–Wang–Wang-2008 and Hwang–Zacharovas-2015)
Suppose $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_m\}$ are multisets of positive integers such that

$$f(q) = \frac{\prod_{j=1}^m [a_j]_q}{\prod_{j=1}^m [b_j]_q} = \sum c_k q^k \in \mathbb{Z}_{\geq 0}[q]$$

.

Let X be a discrete random variable with $\mathbb{P}(X = k) = c_k / f(1)$.
Then the dth cumulant of X is

$$\kappa_d = \frac{B_d}{d} \sum_{j=1}^m (a_j^d - b_j^d)$$

where B_d is the dth Bernoulli number (with $B_1 = \frac{1}{2}$).

Example. This theorem applies to

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) := \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = \frac{q^{b(\lambda)} [n]_q!}{\prod_{c \in \lambda} [h_c]_q}$$

Corollaries of the Distribution Theorem

1. Asymptotic normality also holds for block diagonal skew shapes with aft going to infinity.

2. New proof of asymptotic normality of $[n]_q! = \sum_{w \in S_n} q^{\mathrm{maj}(w)} = \sum_{w \in S_n} q^{\mathrm{inv}(w)}$ due to Feller (1944).

3. New proof of asymptotic normality of q-multinomial coefficients due to Diaconis (1988), Canfield-Jansen-Zeilberger (2011).

4. New proof of asymptotic normality of q-Catalan numbers due to Chen-Wang-Wang(2008).

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4. New proof of asymptotic normality of q-Catalan numbers due to Chen-Wang-Wang(2008).

Question. Using Pak-Panova-Morales's q-hook length formula, can we prove an asymptotic normality for all skew shapes?

Existence Question

Recall $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = \sum b_{\lambda,k} q^k$.

Existence Question. For which λ, k does $b_{\lambda,k} = 0$?

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Cor of Stanley's formula. For every $\lambda \vdash n \geq 1$ there is a unique tableau with minimal major index $b(\lambda)$ and a unique tableau with maximal major index $\binom{n}{2} - b(\lambda')$. These two agree for shapes consisting of one row or one column, and otherwise they are distinct.

Patterns on Tableaux

Example. The min-maj and max-maj tableaux for $(6, 4, 3, 3, 1)$.

1	3	4	11	16	17
2	6	7	15		
5	9	10			
8	13	14			
12					

1	2	3	5	9	13
4	6	10	14		
7	11	15			
8	12	16			
17					

$$b(\lambda) = \sum (i-1)\lambda_i = 23 \qquad \binom{17}{2} - b(\lambda') = 109$$

Existence Question

Recall $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = \sum b_{\lambda,k} q^k$.

Existence Question. For which λ, k does $b_{\lambda,k} = 0$?

Cor of Stanley's formula. The coefficient of $q^{b(\lambda)+1}$ in $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = 0$ if and only if λ is a rectangle.
If λ is a rectangle with more than one row and column, then coefficient of $q^{b(\lambda)+2}$ is 1.

Question. Are there other internal zeros?

Classifying All Nonzero Fake Degrees

Thm.(Billey-Konvalinka-Swanson, 2018+)
For any partition λ which is not a rectangle,

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) := \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)}$$

as no internal zeros. If λ is a rectangle with at least two rows and columns, $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ has exactly one internal zeros at $b(\lambda) + 1$ up to symmetry.

Cor. The irreducible S_n-module indexed by λ appears in the decomposition of the degree k component of the coinvariant algebra if and only if $b_{\lambda,k} > 0$ as characterized above.

Acknowledgment. Our motivation for this project came from a conjecture of Sheila Sundaram's which was solved by Josh Swanson on the zeros of the maj-mod-n generating function on standard Young tableaux.

Exceptional Tableaux

Def. Let $\mathcal{E}(\lambda)$ denote the set of *exceptional* tableaux of shape λ consisting of the following elements:

(i) For all λ, the max-maj tableau for λ.

(ii) If λ is a rectangle, the min-maj tableau for λ.

(iii) If λ is a rectangle with at least two rows and columns, the unique tableau in $\mathrm{SYT}(\lambda)$ with maj equal to $\binom{n}{2} - b(\lambda') - 2$.

Example. $\mathcal{E}(555)$ has the following three elements:

1	2	3
4	5	6
7	8	9

1	2	7
3	5	8
4	6	9

1	4	7
2	5	8
3	6	9

Major Index Increment Map

Proof Outline. We give an explicit map

$$\phi : \mathrm{SYT}(\lambda) - \mathcal{E}(\lambda) \longrightarrow \mathrm{SYT}(\lambda)$$

such that

1. $\mathrm{maj}(\phi(T)) = \mathrm{maj}(T) + 1$,
2. the descent set of $D(T)$ and $D(\phi(T))$ are "close".

Internal Zeros Classification Theorem now follows by starting at the minimal maj tableau in $\mathrm{SYT}(\lambda) - \mathcal{E}(\lambda)$ and applying ϕ recursively until it hits a tableaux in $\mathcal{E}(\lambda)$.

Major Index Increment Map

Pattern Inspired Approach. For each $T \in \mathrm{SYT}(\lambda) - \mathcal{E}(\lambda)$, identify a permutation σ such that $\sigma \cdot T = T'$ is in $SYT(\lambda)$ and $\mathrm{maj}(T') = \mathrm{maj}(T) + 1$.

Example.

1	4
2	5
3	9
6	
7	
8	

$\xrightarrow[(543)]{}$

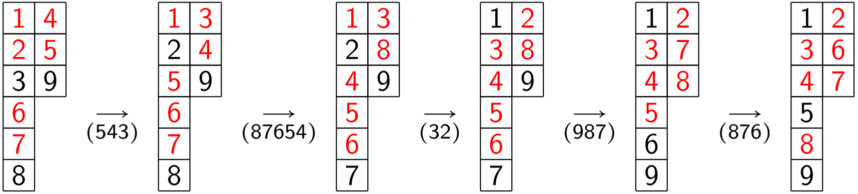
1	3
2	4
5	9
6	
7	
8	

$$D(T) = \{1, 2, 4, 5, 6, 7\} \longrightarrow D(T') = \{1, 3, 4, 5, 6, 7\}$$

Major Index Increment Map

Pattern Inspired Approach. For each $T \in \mathrm{SYT}(\lambda) \setminus maxmaj(\lambda)$, identify a permutation σ such that $\sigma \cdot T = T'$ is in $SYT(\lambda)$ and $\mathrm{maj}(T') = \mathrm{maj}(T) + 1$.

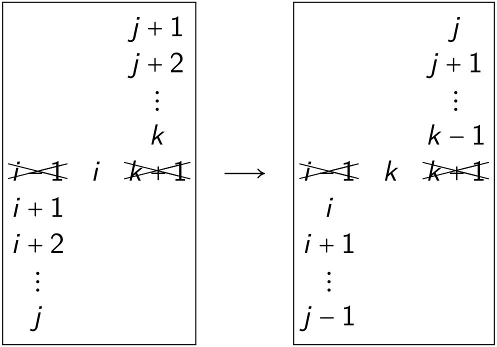
More Examples.



D(T):2 $\longrightarrow$ 3, 7 $\longrightarrow$ 8, 1 $\longrightarrow$ 2, 6 $\longrightarrow$ 7, 5 $\longrightarrow$ 6

Patterns on Tableaux

Rotation Rule. If there exists $i < j < k$, such that the *consecutive values* $[i, k]$ follow the *descent/exclusion pattern*



then the descent set on the left contains $j-1$ and the one on the right contains j, otherwise all other descents are the same.

Patterns on Tableaux

Rotation Rule. If there exists $i = j < k$, such that the values $[i, k]$ follow the descent/exclusion pattern

$$\boxed{\begin{array}{cc} i-1 & i+1 \\ & i+2 \\ & \vdots \\ & k \\ i & \xcancel{k+1} \end{array}} \longrightarrow \boxed{\begin{array}{cc} i-1 & i \\ & i+1 \\ & \vdots \\ & k-1 \\ k & \xcancel{k+1} \end{array}}$$

then the descent set on the left contains $j-1$ and the one on the right contains j, otherwise all other descents are the same.

Patterns on Tableaux

Rotation Rule. If there exists $i < j = k$, such that the values $[i, k]$ follow the descent/exclusion pattern

$$\boxed{\begin{array}{cc} \xcancel{i-1} & i \\ i+1 & \\ \vdots & \\ k-1 & \\ k & k+1 \end{array}} \longrightarrow \boxed{\begin{array}{cc} \xcancel{i-1} & k \\ i & \\ i+1 & \\ \vdots & \\ k-1 & k+1 \end{array}}$$

then the descent set on the left contains $j-1$ and the one on the right contains j, otherwise all other descents are the same.

Patterns on Tableaux

Dual Rotation Rule. If there exists $i < j < k$, such that the values $[i, k]$ follow the descent/exclusion pattern

$$\boxed{\begin{array}{ccccccc} & & & \xcancel{i-1} & i & \cdots & j-1 \\ & & & k & & & \\ j & \cdots & k-1 & \xcancel{k+1} & & & \end{array}} \longrightarrow \boxed{\begin{array}{ccccccc} & & & \xcancel{i-1} & i+1 & \cdots & j \\ & & & i & & & \\ j+1 & \cdots & k & \xcancel{k+1} & & & \end{array}},$$

then the descent set on the left contains $j-1$ and the one on the right contains j, otherwise all other descents are the same.

Patterns on Tableaux

Fact. Almost all standard Young tableaux admit some rotation.

Example. Among the $81,081$ tableaux in $\mathrm{SYT}(5,4,4,2)$, there are only 24 (i.e., 0.03%) on which we cannot apply any rotation rule.

Patterns on Tableaux

Fact. Almost all standard Young tableaux admit some rotation.

Example. Among the $81,081$ tableaux in $\mathrm{SYT}(5,4,4,2)$, there are only 24 (i.e., 0.03%) on which we cannot apply any rotation rule.

Question. What about the tableaux which don't admit any rotation rules?

Block Rules

Five More Block Rules. Adding a descent at 1, plus possibly other mutations.

B1:

1	2	3	4	$\boxed{5}$	$\boxed{16}$	↻	$\boxed{1}$	3	4	5	$\boxed{6}$	$\boxed{15}$
6	7	8	9	10	17		2	7	8	9	10	17
11	12	13	14	15		↻	11	12	13	14	16	

B2:

1	2	3	4	5		$\boxed{1}$	3	4	5	10
6	7	8	9	10	↻	2	7	8	9	13
11	12	13	~~14~~			6	11	12	~~14~~	

Block Rules

B3:

1	2	3	4
5	9		
6	10		
7			
8			

↻

1	3	4	9
2	5		
6	10		
7			
8			

B4:

1	2	7	10
3	5	8	11
4	6	9	12
13	~~14~~		

↻ ↻

1	4	7	11
2	5	8	12
3	6	9	13
10	~~14~~		

Block Rules

$$\text{B5:} \quad \begin{array}{cc} 1 & 2 \\ 3 & 6 \\ 4 & 7 \\ 5 & 8 \\ 9 & \xcancel{10} \end{array} \quad \circlearrowright \quad \begin{array}{cc} \boxed{1} & 6 \\ 2 & 7 \\ 3 & 8 \\ 4 & 9 \\ 5 & \xcancel{10} \end{array}$$

Proof Completion by Cases. Every tableaux which is not exceptional and avoids

$$\begin{array}{cccc} 1 & 2 & \cdots & i \\ i+1 & z+1 & & \\ i+2 & & & \\ \vdots & & & \\ z & & & \end{array}$$

admits a rotation rule. All other non-exceptional tableaux admit a block rule or a rotation rule.

Strong Poset on SYT(λ)

Def. The *Strong SYT Poset* $P(\lambda)$ on either

$$\mathrm{SYT}(\lambda) \setminus \{\mathrm{minmaj}(\lambda), \mathrm{maxmaj}(\lambda)\}$$

if λ is a rectangle with at least two rows and columns, or SYT(λ) otherwise, is the transitive closure of the covering relations given by all applicable rotation rules, block rules, and inverse-transpose block rules, each increasing maj by 1.

Corollary. As a poset, $P(\lambda)$ is ranked according to maj(T) and has a unique minimal and maximal element.

Weak Poset on SYT(λ)

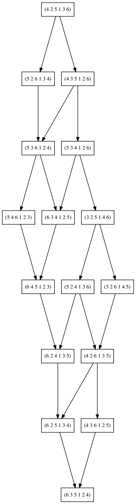
Def. The *Weak SYT Poset* $Q(\lambda)$ on either

$$\mathrm{SYT}(\lambda) \setminus \{\mathrm{minmaj}(\lambda), \mathrm{maxmaj}(\lambda)\}$$

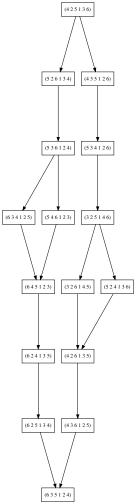
if λ is a rectangle with at least two rows and columns, or SYT(λ) otherwise, is the transitive closure of the relations given by $T < \phi(T)$ and the inverse-transpose of these rules.

Corollary. As a poset, $Q(\lambda)$ is ranked according to maj(T) and has a unique minimal and maximal element.

Strong and Weak Poset on SYT$(3, 2, 1)$



Strong



Weak

Unimodality Question

Conjecture. The polynomial $\mathrm{SYT}^{\mathrm{maj}}(q)$ is unimodal if λ has at least 4 corners. If λ has 3 corners or fewer, then $\mathrm{SYT}^{\mathrm{maj}}(q)$ is unimodal except when λ or λ' is among the following partitions:

1. Any partition of rectangle shape that has more than one row and column.
2. Any partition of the form $(k, 2)$ with $k \geq 4$ and k even.
3. Any partition of the form $(k, 4)$ with $k \geq 6$ and k even.
4. Any partition of the form $(k, 2, 1, 1)$ with $k \geq 2$ and k even.
5. Any partition of the form $(k, 2, 2)$ with $k \geq 6$.
6. Any partition on the list of 40 special exceptions of size at most 28.

Unimodality Question

Special Exceptions.

$$(3,3,2),(4,2,2),(4,4,2),(4,4,1,1),$$
$$(5,3,3),(7,5),(6,2,1,1,1,1),$$
$$(5,5,2),(5,5,1,1),(5,3,2,2),(4,4,3,1),$$
$$(4,4,2,2),(7,3,3),(8,6),(6,6,2),$$
$$(6,6,1,1),(5,5,2,2),(5,3,3,3),(4,4,4,2),$$
$$(11,5),(10,6),(9,7),(7,7,2),$$
$$(7,7,1,1),(6,6,4),(6,6,1,1,1,1),(6,5,5),$$
$$(5,5,3,3),(12,6),(11,7),(10,8),$$
$$(15,5),(14,6),(11,9),(16,6),(12,10),(18,6),$$
$$(14,10),(20,6),(22,6).$$

Local Limit Conjecture

Conjecture. Let $\lambda \vdash n > 25$. Uniformly for all n and for all integers k, we have

$$|\mathbb{P}(X_\lambda[\mathrm{maj}] = k) - N(k; \mu_\lambda, \sigma_\lambda)| = O\left(\frac{1}{\sigma_\lambda \,\mathrm{aft}(\lambda)}\right)$$

where $N(k; \mu_\lambda, \sigma_\lambda)$ is the density function for the normal distribution with mean μ_λ and variance σ_λ.

The conjecture has been verified for $n \leq 50$ and $\mathrm{aft}(\lambda) > 1$.

Up to $n = 50$, the constant $1/9$ works.

At $n = 50$, $1/10$ does not.

Conclusion

P	*E*	*R*	*M*	*U*	*T*	*A*	*T*	*I*	*O*	*N*
A	2									
T	0									
T	1									
E	8									
R										
N										
S										

Many Thanks!