The local structure of semi-sparse permutations

David Bevan
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Based on joint work with **Thomas Selig**.

Permutation Patterns 2018
Dartmouth College, Hanover, NH

9^{th} July 2018

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- What does a large n-permutation with m inversions look like?

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- What does a *typical* large n-permutation with $m(n)$ inversions look like, for some function $m(n)$?

$$m(n) = \rho n^2 \quad \text{— dense}$$

$$m(n) = n^2/\log n \quad \text{— semi-sparse}$$

$$m(n) = n^{3/2} \quad \text{— semi-sparse}$$

$$m(n) = n \log n \quad \text{— semi-sparse}$$

$$m(n) = \alpha n \quad \text{— sparse}$$

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- What properties of an n-permutation with $m(n)$ inversions hold *asymptotically almost surely*?

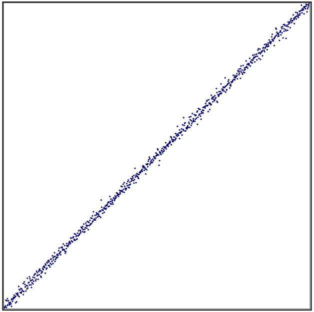
Definition

A property Q holds *asymptotically almost surely* (*a.a.s.*) if $\lim_{n\to\infty} \mathbb{P}[Q] = 1$.

Semi-sparse permutations

A semi-sparse permutation has "few" inversions: $n \ll m \ll n^2$.

- Asymptotic inversion density is zero.



Definition ("y grows faster than x")

We write $x \ll y$ or $y \gg x$ if $\lim_{n\to\infty} x/y = 0$.

Total displacement

What does an n-permutation with m inversions look like?

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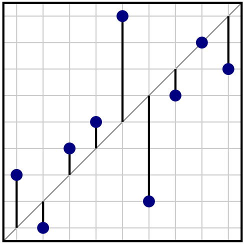
Definition (Knuth)

The *total displacement* of σ is $\mathsf{td}(\sigma) = \sum_i d_i(\sigma) = \sum_i |\sigma(i) - i|$.



$\mathsf{td}(314592687) = 2 + 1 + 1 + 1 + 4 + 4 + 1 + 0 + 2 = 16$

Inversions and total displacement

Proposition (Diaconis & Graham, 1977)

Total displacement satisfies the bounds $\mathsf{inv}(\sigma) \leqslant \mathsf{td}(\sigma) \leqslant 2\mathsf{inv}(\sigma)$.

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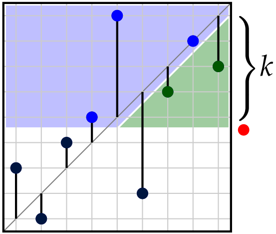
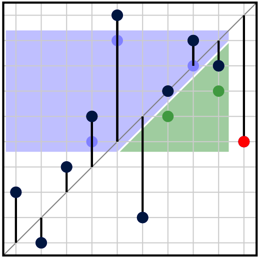
A pictorial proof of the upper bound:



σ



σ'

$\mathsf{inv}(\sigma') = \mathsf{inv}(\sigma) + k$

$\mathsf{td}(\sigma') \leqslant \mathsf{td}(\sigma) + 2k$

What does a semi-sparse permutation look like?

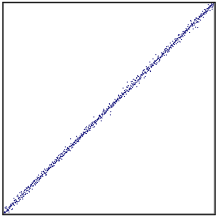
Example ($m = n \log n$)

$$\mathsf{td}(\sigma) = \sum_{i=1}^{n} d_i(\sigma) \leqslant 2n \log n$$

$$\mathbb{E}\big[d_i(\sigma)\big] \leqslant 2 \log n$$

$$\lim_{n\to\infty} \mathbb{P}\big[d_i(\sigma) \gg \log n\big] = 0$$

Almost all the points are close to the main diagonal.



Specific questions: local structure

A semi-sparse permutation has $n \ll m \ll n^2$.

$$\mathcal{P}_{n,m} \;=\; \{\sigma \in S_n : \mathsf{inv}(\sigma) = m\}$$

Select σ_n uniformly from $\mathcal{P}_{n,m}$ and $i < j$ uniformly from $[n]$. Then,

$$\mathbb{P}\big[\sigma_n(i) > \sigma_n(j)\big] \;=\; m/\tbinom{n}{2} \;\to\; 0 \text{ as } n \to \infty.$$

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What does a semi-sparse permutation look like when you zoom in?

- $\lim_{n\to\infty} \mathbb{P}\big[\sigma_n(j) > \sigma_n(j+1)\big]$? (*expected number of descents*)

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- $\lim_{n\to\infty} \mathbb{P}\big[\sigma_n(j) > \sigma_n(j+d)\big]$, for a given d?
- $\lim_{n\to\infty} \mathbb{P}\big[\sigma_n(j+1)\ldots\sigma_n(j+d) \text{ is } \pi\big]$, for given $\pi \in S_d$?

Local structure: consecutive patterns

$$\mathcal{P}_{n,m} = \{\sigma \in S_n : \mathsf{inv}(\sigma) = m\}$$

Proposition

Within $\mathcal{P}_{n,m}$, for any consecutive pattern π and positive $i, j \leqslant n + 1 - |\pi|$,

$$\mathbb{P}\big[\pi \text{ occurs at position } i\big] = \mathbb{P}\big[\pi \text{ occurs at position } j\big].$$

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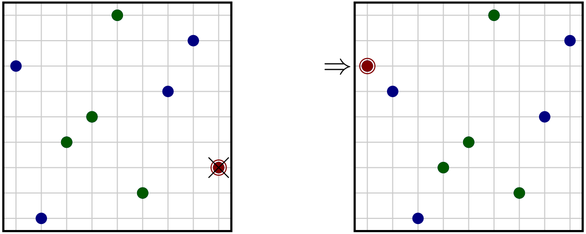
A pictorial proof:



This is a bijection on $\mathcal{P}_{n,m}$ that shifts consecutive patterns right by one.

Results

Theorem

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Results

Theorem (local uniformity)

Suppose $n \ll m \ll n^2/\log^2 n$. Select σ_n uniformly from $\mathcal{P}_{n,m}$.

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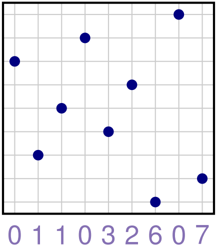
Local-global dichotomy

- Expected number of descents is asymptotically $(n-1)/2$.
- Permutations from $\mathcal{P}_{n,m}$ are locally uniform.
- Local structure reveals nothing about global structure.

Inversion sequences

The *inversion sequence* of σ is (I_j), where $I_j = \big|\{i : i < j \text{ and } \sigma(i) > \sigma(j)\}\big|$.



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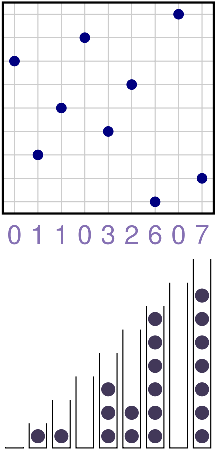
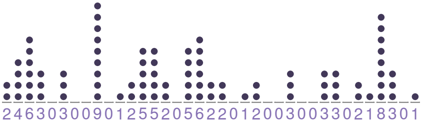


Balls-in-boxes: Sequences with $I_j < j$ whose sum is m are in bijection with n-permutations with m inversions.

Unrestricted balls-in-boxes

Weak compositions of m with n parts, $\mathcal{C}_{n,m}$:



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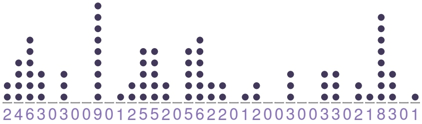

Proposition

If $n \ll m \ll n^2/\log n$, *then for all* $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left[\textit{every box has at most } \; r_{n,m} = \frac{m}{n}(1+\varepsilon)\log n \; \textit{ balls}\right] = 1.$$

A.a.s., no box has more than $(1+\varepsilon)\log n$ times its expected contents.

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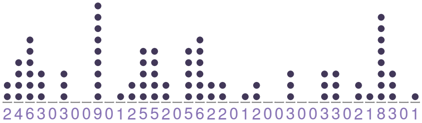

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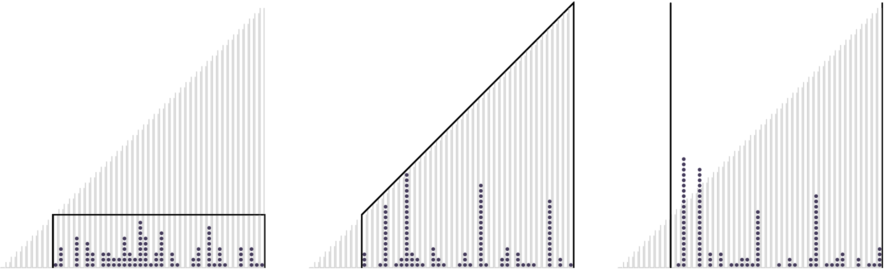
If $\mathcal{C}^{\mathsf{R}}_{n,m}$ is the set of *restricted* compositions with at most $r_{n,m}$ balls per box, then $|\mathcal{C}^{\mathsf{R}}_{n,m}| \sim |\mathcal{C}_{n,m}|$ (that is, $\lim_{n\to\infty} |\mathcal{C}^{\mathsf{R}}_{n,m}|/|\mathcal{C}_{n,m}| = 1$).

Approximating balls-in-boxes



$\mathcal{C}^{\mathsf{R}}_{n,m}$ $\mathcal{C}^{\mathsf{S}}_{n,m}$ $\mathcal{C}_{n,m}$

Since,

$$\mathcal{C}^{\mathsf{R}}_{n,m} \subset \mathcal{C}^{\mathsf{S}}_{n,m} \subset \mathcal{C}_{n,m}$$

and

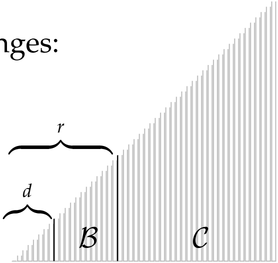
$$|\mathcal{C}^{\mathsf{R}}_{n,m}| \sim |\mathcal{C}_{n,m}|,$$

we also have

$$|\mathcal{C}^{\mathsf{S}}_{n,m}| \sim |\mathcal{C}_{n,m}| = \binom{m+n-1}{m}.$$

Counting

Split boxes into three ranges:



Let $B = \binom{r}{2} - \binom{d}{2}$ be the capacity of $\mathcal{B}$ (boxes $d+1, \ldots, r$).

Let b_k be the number of ways of placing k balls in $\mathcal{B}$.

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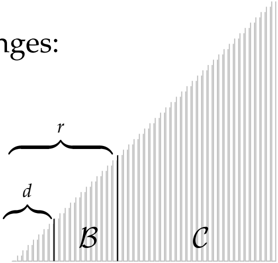

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Number of permutations with the first d boxes empty (increasing):

$$\sum_{k=0}^{B} b_k \left|\mathcal{C}^{\mathsf{S}}_{n-r,m-k}\right| \sim \sum_{k=0}^{B} b_k \left|\mathcal{C}_{n-r,m-k}\right|$$

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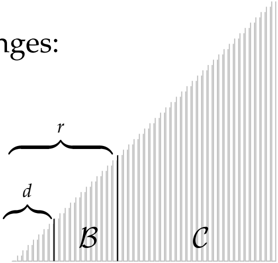

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Number of permutations with the first d boxes full (decreasing):

$$\sum_{k=0}^{B} b_k \left|\mathcal{C}^{\mathsf{S}}_{n-r,m-\binom{d}{2}-k}\right| \sim \sum_{k=0}^{B} b_k \left|\mathcal{C}_{n-r,m-\binom{d}{2}-k}\right|$$

Further results

Theorem (critical window)

Suppose $n \ll m \ll n^2/\log^2 n$. Select σ_n uniformly from $\mathcal{P}_{n,m}$.

1. If $d \sim \alpha m/n$, then

$$0 < \tfrac{1}{1+e^{\alpha}} \leqslant \lim_{n\to\infty} \mathbb{P}\big[\sigma_n(j) > \sigma_n(j+d)\big] \leqslant \tfrac{1}{1+e^{(2-\sqrt{2})\alpha}} < \tfrac{1}{2}.$$

2. If $d \sim \alpha\sqrt{m/n}$, then

$$\mathbb{P}\big[\sigma_n(j+1)\ldots\sigma_n(j+d) \text{ is decreasing}\big] \sim e^{-\alpha^2/4}\frac{1}{d!}.$$

Further questions

Total discrepancy

Is there a simple proof that $\mathsf{td}(\sigma) \geqslant \mathsf{inv}(\sigma)$?

Local uniformity

What about the rest of the semi-sparse range? $n^2/\log^2 n \ll m \ll n^2$
— How can we approximate?

What about dense permutations? $m \sim \alpha n^2$
— Do *permutons* help?

Thanks for listening!