Combinatorial exploration of permutation classes

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Combinatorial classes

A combinatorial class is a set of objects where an object has a notion of size and there are finitely many of each size.

Example

- words
- binary strings
- set partitions
- permutations

Combinatorial specification

A *combinatorial rule* for a combinatorial class $\mathcal{C}$ is a tuple $(\mathcal{C}, \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k\}, \circ)$ such that

$$\mathcal{C} \cong \mathcal{A}_1 \circ \mathcal{A}_2 \circ \cdots \circ \mathcal{A}_k$$

for combinatorial classes $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_k$ and admissible constructor $\circ$.

Combinatorial specification

For non-empty combinatorial classes $\mathcal{C}_1$, $\mathcal{C}_2$, ..., $\mathcal{C}_k$ a *combinatorial specification* is a set of k rules where each $\mathcal{C}_i$ appears on the left of a rule once.

This is equivalent to the definition in Flajolet and Sedgewick [13].

Combinatorial exploration

Two step process.

(1) Expansion - apply *strategies* to combinatorial classes to create a set of rules, we call the *universe*.

(2) Search within the universe for a combinatorial specification.

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(2) Search within the universe for a combinatorial specification.

Our algorithm CombSpecSearcher does this automatically.

Searching for a combinatorial specification

To find a specification, we *prune* the set, that is:

(1) Start with a set of combinatorial rules $\mathcal{U}$

(2) Remove rules $(\mathcal{C}, S, \circ)$ from $\mathcal{U}$ where any combinatorial class in S is not on the left hand side of some rule in $\mathcal{U}$

(3) If no rules removed, terminate, else go to step (1)

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Theorem

Let $\mathcal{U}$ be a set of combinatorial rules. A combinatorial rule $(\mathcal{C}, S, \circ)$ is in a combinatorial specification that is a subset of $\mathcal{U}$ if and only if $(\mathcal{C}, S, \circ)$ is in $\mathcal{U}$ after pruning.

Enumeration of permutation classes

General methods include:

- enumeration schemes, Zeilberger [24] and Vatter [22]
- insertion encoding, Albert, Linton, and Ruškuc [7] and Vatter [23]
- substitution decomposition Albert and Atkinson [1] and Bassino et al. [8]

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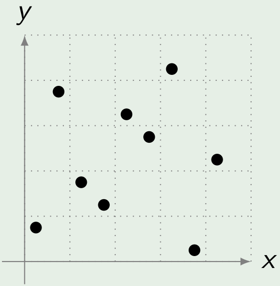
- substitution decomposition Albert and Atkinson [1] and Bassino et al. [8]
- the TileScope algorithm (which uses the CombSpecSearcher algorithm)

Theorem

The methods used to find regular insertion encodings and for the original enumeration schemes as in [24] can be translated into strategies for the TileScope algorithm.

Gridded permutations

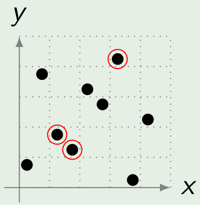
Example



$$\pi = 2^{(0,0)}8^{(0,3)}4^{(1,1)}3^{(1,1)}7^{(2,3)}6^{(2,2)}9^{(3,4)}1^{(3,0)}5^{(4,2)}$$

A gridded permutation π *contains* the gridded permutation σ if there is a subset of points in π that are grid equivalent to σ, otherwise we say π *avoids* σ.

Example



An occurence of $\sigma = 2^{(1,1)}1^{(1,1)}3^{(3,4)}$

A gridded permutation π *avoids* a set of gridded permutations $\mathcal{O}$ if it avoids each σ in $\mathcal{O}$, otherwise we say it contains $\mathcal{O}$.

Define $\mathcal{G}^{(n,m)}$ to be the set of gridded permutations with grid positions in $[0, n-1] \times [0, m-1]$.

$$\mathrm{Av}^{(n,m)}(\mathcal{O}) = \{\pi \in \mathcal{G}^{(n,m)} \mid \pi \text{ avoids } \mathcal{O}\}$$

$$\mathrm{Co}^{(n,m)}(\mathcal{O}) = \{\pi \in \mathcal{G}^{(n,m)} \mid \pi \text{ contains } \mathcal{O}\} = \mathcal{G}^{(n,m)} \setminus \mathrm{Av}^{(n,m)}(\mathcal{O}).$$

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A *tiling* is a triple $\mathcal{T} = ((n,m), \mathcal{O}, \mathcal{R} = \{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k\})$ and

$$\mathrm{Grid}(\mathcal{T}) = \mathrm{Av}^{(n,m)}(\mathcal{O}) \cap \mathrm{Co}^{(n,m)}(\mathcal{R}_1) \cap \cdots \cap \mathrm{Co}^{(n,m)}(\mathcal{R}_k).$$

We call gridded permutations in $\mathcal{O}$ *obstructions* and the sets in $\mathcal{R}$ *requirements*.

Local gridded permutations

A gridded permutation is *local* if all the grid positions are the same. We write π^c to denote a local gridded permutation.

Theorem

Let $\mathcal{C} = \mathrm{Av}(B)$ *be a permutation class and*

$$\mathcal{T} = \left((1,1), \{\sigma^{(0,0)} \mid \sigma \in B\}, \emptyset\right),$$

then $\mathrm{Grid}(\mathcal{T})$ *is in bijection with* $\mathcal{C}$*.*

Av(1342, 3142)

To illustrate the TileScope algorithm we will recover the following result.

Theorem (Kremer [15, 16])

The number of length n permutations that avoid 1342 *and* 3142 *is the n^{th} Schröder number A006318.*

$\mathrm{Av}(1342, 3142)$

To illustrate the TileScope algorithm we will recover the following result.

Theorem (Kremer [15, 16])

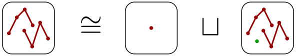
The number of length n permutations that avoid 1342 and 3142 is the n^{th} Schröder number A006318.

What we will show is that this permutation class has the same generating function, namely

$$\frac{3 - x - \sqrt{1 - 6x + x^2}}{2}.$$

$\mathrm{Av}(1342, 3142)$

Every permutation is either empty or contains a point.



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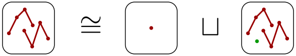
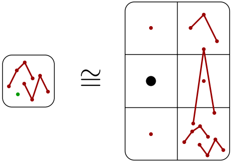


Every non-empty permutation has a leftmost point.



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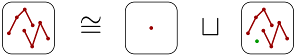

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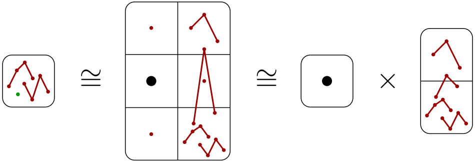
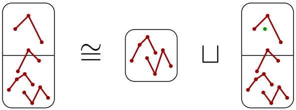
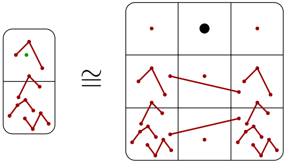


$\mathrm{Av}(1342, 3142)$

The top cell is either empty or contains a point.



The non-empty cell contains a topmost point.



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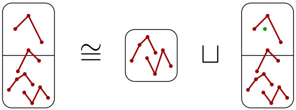

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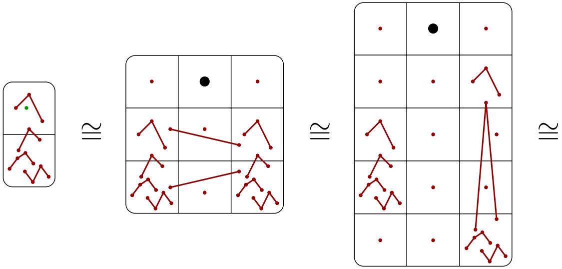

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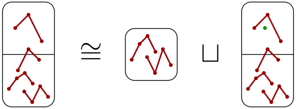

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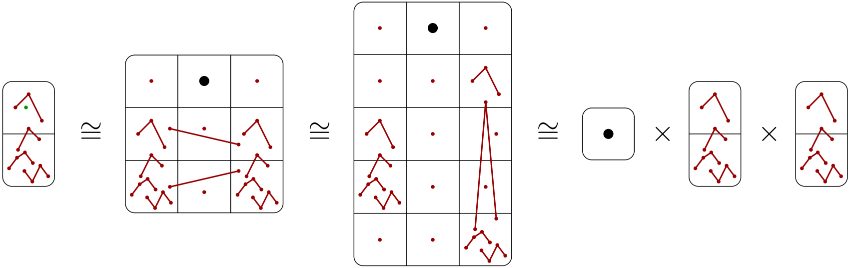
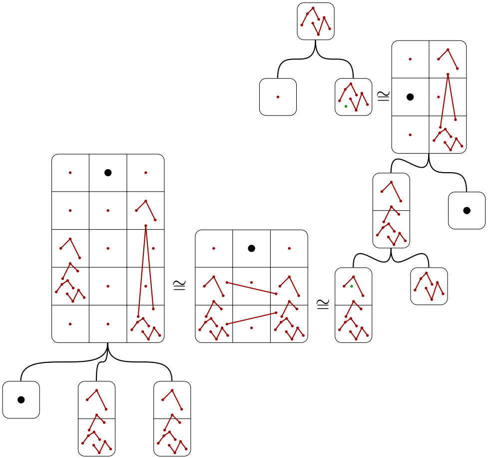


A combinatorial specification for $\mathrm{Av}(1342, 3142)$



$\mathrm{Av}(1342, 3142)$

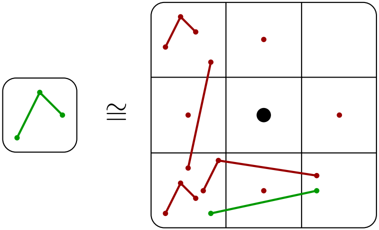
If $F(x)$ is the generating function for $\mathrm{Av}(1342, 3142)$ then the combinatorial specification implies

$$F(x) = 1 + x\left(F(x) + x\left(\frac{F(x)-1}{x}\right)^2\right).$$

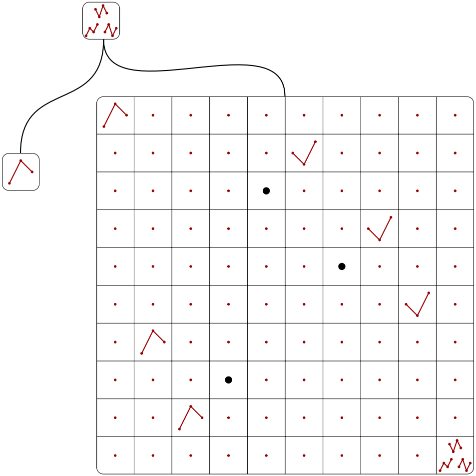
Solving gives

$$F(x) = \frac{3 - x - \sqrt{1 - 6x + x^2}}{2}.$$

Consider a permutation that contains 132. Place the leftmost point that is a 3 in an occurrence of 132.



Av(1324, 2413, 3142)



Elementary permutation class

If all the obstructions and requirements of a tiling are local and it is fully separated it is called *elementary*.

Any permutation class is *elementary* if it can be described by a disjoint union of elementary tilings.

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Theorem (Homberger and Vatter [14])

Every polynomial permutation class is elementary.

Elementary permutation classes

If a permutation class is elementary, then adding a pattern to its basis results in an elementary permutation class.

Let $S(\mathcal{T})$ be the set of underlying permutations of the gridded permutations in the set $\mathrm{Grid}(\mathcal{T})$.

Theorem

Let $\mathcal{C} = \mathrm{Av}(B)$ be an elementary permutation class, given by

$$\mathcal{C} = S(\mathcal{T}_1) \sqcup S(\mathcal{T}_2) \sqcup \cdots S(\mathcal{T}_k)$$

For any pattern σ in $\mathcal{S}$ the permutation class $\mathrm{Av}(B \cup \{\sigma\})$ is elementary if there is no $S(\mathcal{T}_i)$ with $\mathrm{Av}(B \cup \{\sigma\}) \subseteq S(\mathcal{T}_i)$.

We experimentally search for elementary permutation classes whose bases have length 4 patterns.

$\|B\|$	non-symmetric	minimal elementary bases	total number of elementary bases	total number of non-elementary bases	non-regular insertion-encoding and non-elementary
12	342424	0	342424	0	0
11	316950	0	316949	1	0
10	249624	0	249611	13	0
9	166786	0	166717	69	0
8	94427	0	94196	231	3
7	44767	0	44260	507	28
6	17728	5	16933	795	108
5	5733	44	4890	843	222
4	1524	334	903	621	244
3	317	38	44	273	143
2	56	1	1	55	43
1	7	0	0	7	7
total	-	422	-	3416	798

Table: The successes for bases consisting of length 4 patterns with elementary point placement strategies.

Success with point placement strategies

Our success allowing the full power of combinatorial specifications and our strategies.

$\|B\|$	non-insertion-encodable and non-elementary	successes	number of bases remaining
8	3	3	0
7	28	38	0
6	108	107	1
5	222	217	5
4	244	228	16
3	143	115	27
2	43	21	22
1	7	0	7
total	798	720	78

Table: The successes for bases consisting of length 4 patterns with point placement strategies.

The 2x4 successes

B	OEIS sequence	reference to first enumeration
1234, 3412	A165525	Albert, Atkinson, and Brignall [3]
1243, 2143	A155069	Kremer [15, 16]
1243, 2413	A165538	Albert, Atkinson, and Vatter [5]
1243, 2431	A165534	Pantone [21]
1243, 3412	A165529	Albert, Atkinson, and Brignall [3]
1243, 4231	A165526	Albert, Atkinson, and Brignall [3]
1324, 2143	A032351	Bóna [11]
1324, 2413	A032351	Bóna [11]
1324, 3412	A165527	Albert, Atkinson, and Brignall [2]
1324, 4231	A165528	Albert, Atkinson, and Vatter [4]
1342, 2413	A165541	Albert, Atkinson, and Vatter [5]
1342, 2143	A109033	Le [18]
1342, 2413	A165541	Albert, Atkinson, and Vatter [5]
1342, 2431	A032351	Bóna [11]
1342, 3142	A155069	Kremer [15, 16]
1342, 3241	A032351	Bóna [11]
1432, 2413	A032351	Bóna [11]
1432, 3412	A047849	Kremer and Shiu [17]
2413, 3142	A155069	Kremer [15, 16]

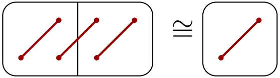
Table: Bases consisting of two length 4 patterns that succeed with point placement strategies. They are listed with their OEIS sequence and the reference to the paper that first enumerated it.

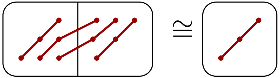
Other notable successes

$\mathrm{Av}(1324, 24153, 31524, 426153)$ - these permutations index the DBI Schubert varieties, Albert and Brignall [6]

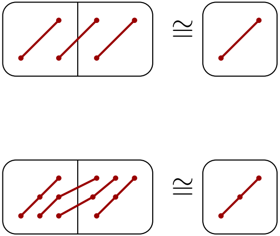
$\mathrm{Av}(1243, 1342, 1423, 1432, 2143, 35142, 354162, 461325, 465132)$ - the class of permutations containing at most one copy of 132, Bóna [10]

Fusion





Fusion



To enumerate this we need a catalytic variable to track the region that fuses.

Our success allowing the fusion strategy.

$|B|$	number of bases remaining	success with fusion	enumerated
6	1	1	1
5	5	5	5
4	16	16	11
3	27	27	13
2	22	18	7
1	7	2	0
total	78	69	37

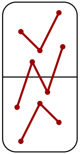
Table: The successes for bases consisting of length 4 patterns with fusion.

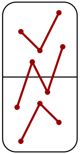
The 2x4 fusion successes

B	OEIS sequence	reference to first enumeration
1243, 2134	A164651	Callan [12] and Le [18]
1243, 2341	A165536	Miner [19]
1324, 1342	A155069	Kremer [15, 16]
1342, 2341	A155069	Kremer [15, 16]
1342, 3124	A164651	Callan [12] and Le [18]
1342, 4123	A165533	Miner [19]
2143, 2413	A165546	Miner and Pantone [20]

Table: Bases consisting of two length 4 patterns that succeed with fusion strategies. They are listed with their OEIS sequence and the reference to the paper that first enumerated it.

Other notable successes





This tiling was used in Bevan et al. [9] to improve the lower bound of the growth rate of $\mathrm{Av}(1324)$ to 10.271.

Conclusion

The TileScope algorithm has been able to rederive the results of many articles, and derive many new results under one unified method.

Please send bases you would like to be run to: bit.ly/basisrequests

The CombSpecSearcher algorithm is a general purpose algorithm for performing combinatorial exploration. It is ready to be applied to other combinatorial classes.

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