Consecutive Patterns in Inversion Sequences

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Joint work with Sergi Elizalde

Inversion Sequences

An inversion sequence of length n is an integer sequence $e = e_1e_2\cdots e_n$ such that $0 \leq e_i < i$. The set of all inversion sequences e of length n is denoted by $\mathbf{I}_n$.

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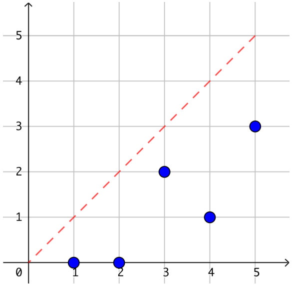

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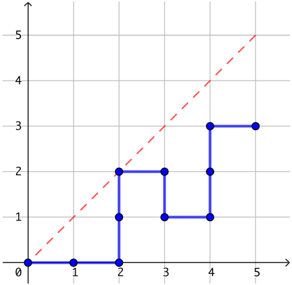

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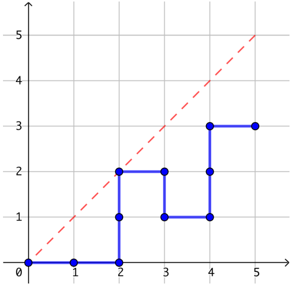


We can encode permutations as inversion sequences by defining a bijection $\Theta : S_n \to \mathbf{I}_n$, where $\Theta(\pi) = e = e_1 e_2 \cdots e_n$ is and

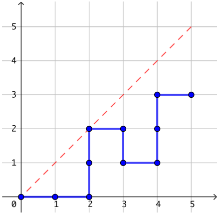
$$e_i = |\{j : j < i \text{ and } \pi_j > \pi_i\}|.$$

For instance, $\Theta(35142) = 00213$.

Classical Patterns

- The reduction of $e \in \mathbf{I}_n$ is the word obtained by replacing the ith smallest entry of e with $i-1$.
- We say e contains the (classical) pattern $p = p_1p_2\cdots p_l$ if there exist $i_1 < i_2 < \cdots < i_l$ such that the reduction of $e_{i_1}e_{i_2}\cdots e_{i_l}$ is p. Otherwise, e avoids p.

Example. $e = 00213$ contains 012 and 001. But it avoids 201 and 110.



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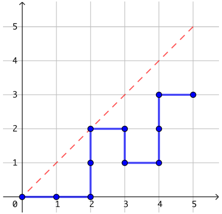


We denote $\mathbf{I}_n(p) = \{e \in \mathbf{I}_n : e \text{ avoids } p\}$. For instance, $\mathbf{I}_3(001) = \{000, 010, 011, 012\}$.

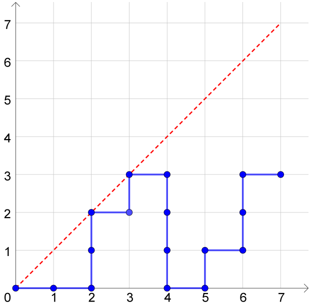
The avoidance sequences $|\mathbf{I}_n(p)|$ have been studied by Corteel, Martinez, Savage and Weselcouch. Independently by Mansour and Shattuck.

$|\mathbf{I}_n(012)|$: Boolean permut.
$|\mathbf{I}_n(021)|$: Large Schröder numb.
$|\mathbf{I}_n(000)|$: Euler up/down numb.
$|\mathbf{I}_n(011)|$: Bell numb.

Consecutive Patterns

We say $e \in \mathbf{I}_n$ contains the (consecutive) pattern $p = \underline{p_1 p_2 \cdots p_l}$ if there exists i such that $e_i e_{i+1} \cdots e_{i+l-1}$ has reduction p. Otherwise, we say e avoids p.

Example. $e = 0023013$ avoids $\underline{000}$ and $\underline{010}$, but it contains $\underline{012}$ and $\underline{120}$.



We consider the problem of determining the sequence $|\mathbf{I}_n(p)|$ for consecutive patterns $p = \underline{p_1 p_2 \cdots p_l}$.

Our Results for Consecutive Patterns of Length 3

We denote $\mathbf{I}_{n,k}(p) = \{e \in \mathbf{I}_n(p) : e_n = k\}$.

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Pattern p	in the OEIS?	Observations about $\lvert\mathbf{I}_n(p)\rvert$
$\underline{012}$	A049774	Counts $\lvert S_n(\underline{321})\rvert$
$\underline{021}$	A071075	Counts $\lvert S_n(\underline{132}4)\rvert$
$\underline{102}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - \sum_{j\geq 1} j \cdot \lvert\mathbf{I}_{n-2,j}(p)\rvert$
$\underline{120}$	A200404	Counts $\lvert S_n(\underline{143}2)\rvert$
$\underline{201}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - k\sum_{j>k} \lvert\mathbf{I}_{n-2,j}(p)\rvert$
$\underline{210}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - \sum_{m>k}\sum_{j>m}\sum_{i\leq j} \lvert\mathbf{I}_{n-3,i}(p)\rvert$

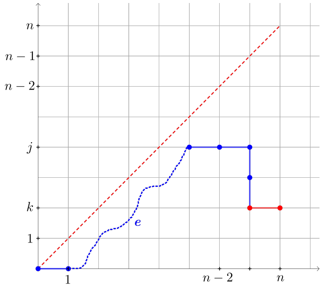
Pattern p	in the OEIS?	Observations about $\lvert\mathbf{I}_n(p)\rvert$
$\underline{000}$	A052169	Equals $\frac{(n+1)!-d_{n+1}}{n}$, where d_n is the number of derangements of [n]
$\underline{001}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - \sum_{j<k} \lvert\mathbf{I}_{n-2,j}(p)\rvert$
$\underline{010}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - (n-2-k)\lvert\mathbf{I}_{n-2,k}(p)\rvert$
$\underline{011}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - \sum_{j<k} \lvert\mathbf{I}_{n-2,j}(p)\rvert$ if $k \neq n-1$, and $\lvert\mathbf{I}_{n,n-1}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert$
$\underline{100}, \underline{110}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - \sum_{j>k} \lvert\mathbf{I}_{n-2,j}(p)\rvert$
$\underline{101}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - k\lvert\mathbf{I}_{n-2,k}(p)\rvert$

Recurrences

A recurrence for the pattern $p = \underline{110}$:

$$|\mathbf{I}_{n,k}(p)| = |\mathbf{I}_{n-1}(p)| - \sum_{j>k} |\mathbf{I}_{n-2,j}(p)|,$$

with $|\mathbf{I}_0(p)| = 1$ and $\mathbf{I}_n(p) = \sum_{j=0}^{n-1} |\mathbf{I}_{n,j}(p)|$.



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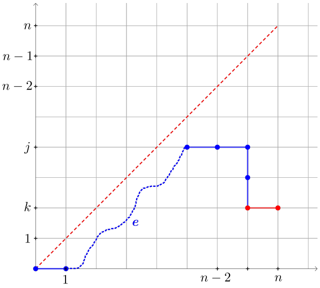

For the pattern $p = \underline{000}$:

- Recurrence. $|\mathbf{I}_n(p)| = (n-1)\,|\mathbf{I}_{n-1}(p)| + (n-2)\,|\mathbf{I}_{n-2}(p)|$.
- EGF. $2 - \frac{1}{e}\mathsf{Ei}(1) + \frac{x\,e^x - 1}{e^x(1-x)} + \log\frac{1}{1-x} + \frac{1}{e}\mathsf{Ei}(1-x)$, with $\mathsf{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t}\,dt$.

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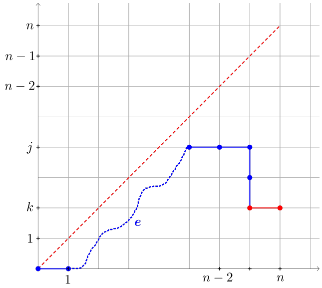

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- Closed form. $|\mathbf{I}_n(p)| = \frac{(n+1)!}{n}\left(1 - \sum_{j=0}^{n+1} \frac{(-1)^j}{j!}\right)$.

Equivalence Between Patterns

We say two consecutive patterns p and p' are:

- Wilf equivalent, denoted $p \sim p'$, if for all n,

$$|\mathbf{I}_n(p)| = |\mathbf{I}_n(p')|.$$

- Strongly Wilf equivalent, denoted $p \overset{s}{\sim} p'$, if for all n and m,

$$|\{e \in \mathbf{I}_n : e \text{ has } m \text{ occur. of } p\}| = |\{e \in \mathbf{I}_n : e \text{ has } m \text{ occur. of } p'\}|.$$

- If the above condition holds for any set of positions for the m occurrences, then p and p' are super strongly Wilf equivalent, denoted $p \overset{ss}{\sim} p'$.

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Example. $e = 0023013$ has two occurrences of the pattern $\underline{012}$; in positions 2 and 5.

The Patterns $\underline{100}$ and $\underline{110}$ are Equivalent

Proof 1. $|\mathbf{I}_n(\underline{100})|$ and $|\mathbf{I}_n(\underline{110})|$ satisfy the same recurrence, so $\underline{100} \sim \underline{110}$.

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Proof 2. (Sketch of $\underline{100} \overset{ss}{\sim} \underline{110}$)

1. For $p = \underline{100}, \underline{110}$, define

$$\mathsf{Em}(p, e) = \{i : e_i e_{i+1} e_{i+2} \text{ is an occur. of } p\}.$$

2. Construct a bijection

$$\{e \in \mathbf{I}_n : \mathsf{Em}(\underline{100}, e) \supset S\} \longleftrightarrow \{e \in \mathbf{I}_n : \mathsf{Em}(\underline{110}, e) \supset S\}.$$

3. Use inclusion–exclusion to see

$$|\{e \in \mathbf{I}_n : \mathsf{Em}(\underline{100}, e) = S\}| = |\{e \in \mathbf{I}_n : \mathsf{Em}(\underline{110}, e) = S\}|.$$

This is the only equivalence between consecutive patterns of length 3.

Longer Consecutive Patterns

Theorem. A complete list of equivalences between consecutive patterns of length 4 is as follows:

- $\underline{2013} \stackrel{ss}{\sim} \underline{2103}$
- $\underline{3012} \stackrel{ss}{\sim} \underline{3102}$
- $\underline{0100} \stackrel{ss}{\sim} \underline{0110}$
- $\underline{1000} \stackrel{ss}{\sim} \underline{1110}$
- $\underline{1101} \stackrel{ss}{\sim} \underline{1001} \stackrel{ss}{\sim} \underline{1011}$
- $\underline{0021} \stackrel{ss}{\sim} \underline{0121}$
- $\underline{1002} \stackrel{ss}{\sim} \underline{1102} \stackrel{ss}{\sim} \underline{1012}$
- $\underline{2001} \stackrel{ss}{\sim} \underline{2101} \stackrel{ss}{\sim} \underline{2011} \stackrel{ss}{\sim} \underline{2201}$
- $\underline{1200} \stackrel{ss}{\sim} \underline{1210} \stackrel{ss}{\sim} \underline{1220}$
- $\underline{2100} \stackrel{ss}{\sim} \underline{2210}$
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Conjecture. Two consecutive (inversion sequence) patterns of length m are strongly Wilf equivalent if and only if they are Wilf equivalent.

Patterns of Relations

Define the set $\mathbf{I}_n\left(R_1, R_2\right)_c$, with $R_i \in \{\leq, \geq, <, >, =, \neq\}$, consisting of inversion sequences $e \in \mathbf{I}_n$ for which there is no i such that $e_i R_1 e_{i+1}$ and $e_{i+1} R_2 e_{i+2}$.

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Theorem. A complete list of equivalences between consecutive patterns of relations of length 3 is as follows:

- $(\geq, \geq)_c \overset{ss}{\sim} (<, <)_c$
- $(\geq, <)_c \overset{ss}{\sim} (<, \geq)_c \overset{s}{\sim} (\neq, \geq)_c$
- $(\geq, >)_c \overset{ss}{\sim} (>, \geq)_c$
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Reproving a result of Baxter–Pudwell:

Corollary. The generalized permutation patterns $\underline{124}3$ and $\underline{421}3$ are Wilf equivalent.

Our Results for Patterns of Relations

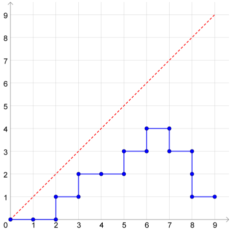
Connections to other combinatorial objects:

Pattern p	$\lvert\mathbf{I}_n(p)\rvert$ in the OEIS as	OEIS description
$(\leq,\geq)_c$	A000027	n
$(\leq,>)_c$	A000108	C_n: Catalan numbers
$(\leq,\neq)_c$	A040000	$1,2,2,\ldots$ (constant 2 for $n>1$)
$(\geq,\leq)_c$	A000045	F_{n+2}: $(n+2)$th Fibonacci number
$(\geq,\geq)_c \overset{ss}{\sim} (<,<)_c$	A049774	$\lvert S_n(\underline{321})\rvert$
$(\geq,<)_c \overset{ss}{\sim} (<,\geq)_c \overset{s}{\sim} (\neq,\geq)_c$	A000079	2^{n-1}
$(\geq,>)_c \overset{ss}{\sim} (>,\geq)_c$	A200403	$\lvert S_n(\underline{124}3)\rvert$
$(\geq,\neq)_c$	A000124	Central polygonal numbers (lazy caterer's sequence)
$(>,\leq)_c$	A071356	Motzkin paths of length n with up and level steps coming in two colors
$(=,=)_c$	A052169	$\frac{(n+1)!-d_{n+1}}{n}$
$(\neq,\leq)_c$	A000071	$F_{n+2}-1$, where F_{n+2} is the $(n+2)$th Fibonacci number
$(\neq,=)_c$	A000522	Number of 01−avoiding rook monoids
$(\neq,\neq)_c$	A000085	Number of involutions of $[n]$

The Pattern $\mathbf{I}_n(>,\leq)_c$

Observation. (Martinez–Savage)
$e \in \mathbf{I}_n(>,\leq)_c$ iff there exists k such that

$e_1 \leq e_2 \leq \cdots \leq e_k > e_{k+1} > e_n$.



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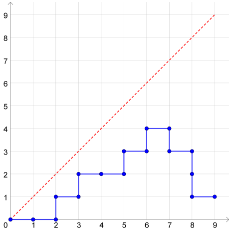

Theorem. The sequence $|I_n\left(>, \leq\right)_c|$ has OGF,

$$A(x) = \frac{1 - 2x - \sqrt{1 - 4x - 4x^2}}{4x^2}.$$

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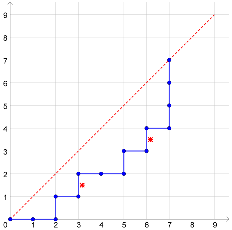

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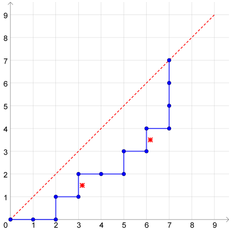

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Moreover:

- Let $\text{dist}(e)$ be the number of distinct elements of e. Then $\sum_{e \in \mathbf{I}_n(>, \leq)} y^{\text{dist}(e)}$ is unimodal and palindromic (conjectured by Martinez–Savage).
- We can count all variations of unimodal inversion sequences (also according to $\text{dist}(e)$).

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