

Wilf-collapse in permutation classes

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Permutation Patterns 2018,
Dartmouth College



Definitions

Let \mathcal{C} be a permutation class.

- ▶ The *enumeration sequence* for \mathcal{C} is just $(c_n)_{n \geq 0}$ where c_n is the number of permutations of size n in \mathcal{C}
- ▶ For $\pi \in \mathcal{C}$, the *principal subclass*, $\mathcal{C}_\pi = \mathcal{C} \cap Av(\pi)$ of \mathcal{C} consists of all those permutations in \mathcal{C} that *do not* have π as a subpermutation.

Wilf-equivalence and Wilf-collapse

- ▶ Two permutation classes \mathcal{C} and \mathcal{D} are *Wilf-equivalent* if $c_n = d_n$ **for all** n .
- ▶ If \mathcal{C} is a class, and $\pi, \tau \in \mathcal{C}$ then π is *Wilf-equivalent* to τ relative to \mathcal{C} ($\pi \sim_{\mathcal{C}} \tau$) if the principal subclasses \mathcal{C}_{π} and \mathcal{C}_{τ} are Wilf-equivalent.
- ▶ \mathcal{C} exhibits a *Wilf-collapse* if the number, w_n , of $\sim_{\mathcal{C}}$ -equivalence classes for permutations of size n in \mathcal{C} is small compared to c_n , i.e., $w_n = o(c_n)$.

Some prior work

- ▶ (2015, MA and Mathilde Bouvel) *The class $\text{Av}(312)$ has an exponential Wilf-collapse. In fact*

$$w_n = o\left(\left(\frac{5}{8}\right)^n c_n\right).$$

This bound is not tight.

- ▶ (2017, MA and Jinge Li) *Any class with two basis elements of size three has an (exponential if possible) Wilf-collapse.*
- ▶ In the first case we have a conjecturally complete description of \sim_c , in the second case a complete description.
- ▶ If there is a theme it is: *global symmetry applied locally often results in Wilf-equivalence.*

A dream and a plan

- ▶ The dream is to show that many classes exhibit Wilf-collapse.
- ▶ The plan is to find sufficient conditions for Wilf-equivalence that apply to a wide variety of classes.
- ▶ For example if we could show that: *except for exponentially few permutations in \mathcal{C} their $\sim_{\mathcal{C}}$ -equivalence classes are exponentially large*, that gets an exponential Wilf-collapse.

A particular case

- ▶ Consider classes \mathcal{C} which contain only a finite set, \mathcal{A} , of sum-indecomposable permutations.
- ▶ The elements of \mathcal{C} are naturally represented as words over the alphabet \mathcal{A} via their sum-decomposition.
- ▶ This is a bijection between \mathcal{C} and \mathcal{A}^* if \mathcal{C} is sum-closed.
- ▶ The subpermutation ordering on \mathcal{C} is a refinement of the subword ordering on \mathcal{A}^* which can be thought of as taking the transitive closure of it and certain basic relations $A \leq a$ where $A = a_1 a_2 \cdots a_k \in \mathcal{A}^*$, $a \in \mathcal{A}$, and

$$a_1 \oplus a_2 \oplus \cdots \oplus a_k \leq a$$

as permutations.

Gekko's principle

If,

- ▶ \mathcal{A} is a set of sum-indecomposable permutations.
- ▶ $A \in \mathcal{A}^*$ and $B = bC \in \mathcal{A}^*$
- ▶ $A = PS$ where P is the maximal prefix of A with $P \leq b$.

Then,

$$A \leq B \iff S \leq C.$$

The basic idea

- ▶ Let $\mathcal{C} = \oplus \mathcal{A}$
- ▶ Let a be a maximal element of \mathcal{A} .
- ▶ Consider two words:

$$\begin{aligned} A &= X \ a(\bullet) a \ Y \ a(\bullet\bullet) a \ Z \\ B &= X \ a(\bullet\bullet) a \ Y \ a(\bullet) a \ Z \end{aligned}$$

- ▶ Claim: $A \sim_{\mathcal{C}} B$.
- ▶ Reason: If $A \leq W$ witness this greedily. Now exchange the “part of W that matches the (\bullet) ” and the “part of W that matches the $(\bullet\bullet)$ ” to produce a W' with $B \leq W'$ and note this is a bijection.
- ▶ The key point is that because of the maximality of a it can only be matched by literally some other a so it forces a break in the word.

Why this helps

- ▶ Words over a finite weighted alphabet have good statistical behaviour (an instance of the *supercritical* case discussed in F&S).
- ▶ In particular, in a word of weight n , the number of occurrences of any fixed subword (e.g., $a(\bullet)a$) is linear (in n) except with exponentially small probability.
- ▶ From previous slide, any two words which differ only in the order of their $a(\bullet)a$ or $a(\bullet\bullet)a$ factors are Wilf-equivalent.
- ▶ And

$$\binom{(c+d)n}{cn}$$

grows exponentially.

A slight generalisation

- ▶ Let \mathcal{C} be a sum-closed class whose sum-indecomposables are \mathcal{A} , with generating function $A(t)$.
- ▶ Suppose that $A(t) = 1$ has a solution r strictly inside its radius of convergence.
- ▶ Suppose there exist $a, b \in \mathcal{A}$ such that $ab \not\leq c$ for all $c \in \mathcal{A}$.

Theorem

\mathcal{C} has an exponential Wilf-collapse.

Note: None of these conditions apply to $\mathcal{C} = \text{Av}(312)$ but the conclusion holds.

Is sum-closure needed?

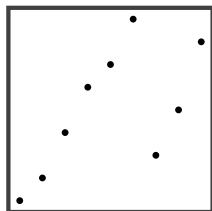
- ▶ No.
- ▶ In the original setting above it is the case that for some $k \geq 0$ and some $c \geq 1$,

$$c_n = \Theta(n^k c^n).$$

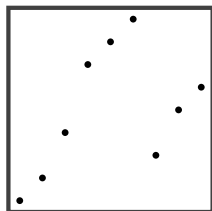
- ▶ A finite state automaton recognises \mathcal{C} .
- ▶ Almost all paths in this automaton of weight n visit a special kind of state exactly $k + 1$ times.
- ▶ Those typical paths belong to exponentially large Wilf-classes.
- ▶ It's not clear that the residue is exponentially small so we do not yet have an exponential Wilf-collapse.
- ▶ This argument *may* generalise to the supercritical case provided that \mathcal{C} is finitely based.

A word of warning

In $Av(321)$ the *shortest* pair of permutations beginning with their two least elements that appear to be Wilf-equivalent¹ but not for reasons of symmetry are:



$12 \oplus 2457136$



$12 \oplus 2567134$

However, it has not been ruled out that $\alpha \oplus 2457136 \sim_{Av(321)} \alpha \oplus 2567134$, for all α , which would be a positive indication of eventual Wilf-collapse in $Av(321)$.

¹checked to size 18

Lessons learned

- ▶ A greedy method for checking for involvement lets one apply ‘global’ Wilf-equivalences to ‘local’ pieces.
- ▶ This generally demonstrates an exponential Wilf-collapse.
- ▶ In retrospect, arguments of MA and Mathilde Bouvel in the class of 312-avoiding permutations can also be cast in this light.
- ▶ We have many other examples of lesser and greater specificity – but they all fit into that framework.
- ▶ In *every* example where $c_n \rightarrow \infty$ **and** we can compute or estimate w_n it is the case that $w_n = o(c_n)$.