Welcome to Dartmouth!

We're so pleased to welcome you to Dartmouth College and to Hanover, New Hampshire for Permutation Patterns 2018. We hope that you enjoy the conference and all the College and the Upper Valley have to offer. We encourage you to take some time this week to explore the beautiful and historic Dartmouth campus—see the *Things to Do* section for a few suggestions.

If you need anything during your stay, please don't hesitate to reach out to one of our local organizers or our department administrators, Tracy Moloney and Amy Potter, all of whom will have special nametags with a black background.

Whether you came from down the street or across the globe, thank you for trekking your way here through the woods of New Hampshire and we hope you have a fun and productive week!

Sincerely yours,
Jay Pantone
Organizing Committee chair

Sponsors

Permutation Patterns 2018 is generously supported by the Dean of the Faculty Office at Dartmouth College through the Conant 1879 Memorial Lectureship and the Robert 1931 and Ruth Fraser Funds, by the Department of Mathematics at Dartmouth College, by NSF grant DMS-1764236, and by NSA grant H98230-18-1-0231.

Important Numbers

Mathematics Department	(603) 646-2415, Mon–Fri, 8am–4pm
Jay Pantone	(561) 635-2392
Tracy Moloney	(603) 646-3723, Kemeny 102D
Safety and Security	(603) 646-4000
Pin code for Berry and Bildner Halls	6188#

Dartmouth Hitchcock Medical Center is located 3 miles from campus. The main number is (603) 650-5000. You should call 911 if you have a medical emergency.

Campus Dining

The main campus dining hall is *'53 Commons*, known to Dartmouth students as *Foco*. '53 Commons has all-you-can-eat buffet-style service and offers quite a wide variety of options. They have a dedicated meat-free vegetarian kitchen and a dairy-free kosher kitchen. Their hours are

	Monday – Saturday	**Sunday**
Breakfast	7:00am – 10:30am	7:00am – 2:30pm
Lunch	11:00am – 2:30pm	
Dinner	5:00pm – 8:30pm	5:00pm – 8:30pm

Note that '53 Commons does not close between breakfast and lunch on Sundays.

Participants staying in the dormitories have received a meal card as part of their accommodation costs. Breakfast, lunch, and dinner are included Monday–Friday for any days for which you're spending the night (for example, if you're staying in the dorms Thursday night, then your card has meals for Thursday). The cards do not include dinner on Tuesday (the night of the conference banquet).

Those without meal cards can pay at the door: $7.75 for breakfast, $10.75 for lunch, and $14.95 for dinner.

Local Restaurants

There are a surprising number of restaurants within walking distance. A few that we can recommend are Base Camp Cafe (Nepalese cuisine), C&A Pizza, Candela Tapas, Canoe Club, Dirt Cowboy Cafe, Jewel of India, Lou's Restaurant & Bakery, Molly's Restaurant & Bar, Morano Gelato, Murphy's on the Green, Orient Restaurant, Pine (quite expensive), Ramunto's Brick & Brew Pizzeria, Salt Hill Pub, Sushi-ya, and Tuk Tuk Thai.

Feel free to ask any of the local organizers for recommendations!

Wifi

The *Dartmouth Public* wifi network is open for any participants to use throughout campus. *Eduroam* is also available for those who already have access via their institutions.

Parking

Parking passes are required to park on campus on weekdays from 6:00am to 5:00pm. All participants must park in Dewey Lot on the North edge of campus. There is a map of campus later in this program.

Gym Passes

The main Dartmouth gym is located at 16 East Wheelock Street. It has a pool, exercise machines, squash courts, and an indoor track. You can buy day passes at the entrance to the gym for $10.

Things to Do

See the Orozco Murals in Baker Library
José Clemente Orozco (1883–1959) was a Mexican social realist painter. One of his most famous murals is the *Epic of American Civilization*, which covers 3200 square feet in the basement of Baker Library.

Visit the Life Sciences Greenhouse
The Life Sciences Greenhouse has an extensive and varied plant collection with a wide range of diversity, utility, and beauty. Among the most popular and diverse of their collections is the Brout Orchid Collection, home to about a thousand orchids of many species and hybrids. The orchid collection is in two rooms. There are three other rooms open to the public: a tropical room, a sub-tropical room, and a xeric room housing cacti and succulents. The Life Sciences building is next to the Dewey parking lot.

Go for a walk
Dartmouth's campus is full of historic landmarks such as Bartlett Tower, a statue of

Robert Frost (who was a Dartmouth student for a very brief time in the 1890s), and Webster Cottage.

Go for a hike

Hiking maps are available at the Dartmouth Outing Club in Robinson Hall.

Explore Vermont

Notable Vermont attractions include the Queeche gorge, Woodstock and the Long Trail Brewery.

Welcome Reception

Sunday, 5:30pm–7:30pm, Occom Commons / Goldstein Hall
Please join us for hamburgers, hotdogs, and refreshments at Occom Commons in Goldstein Hall. Goldstein Hall is directly adjacent to the conference dormitories, Berry and Bildner Halls.

Poster Session

Monday, 5:00pm–7:00pm, Occom Commons / Goldstein Hall
The poster session will be held on Monday evening. There will be hors d'oeuvres and refreshments. The following posters will be presented.

- **K block set partition patterns and statistics**, *Amrita Acharyya*
- **Some relations on prefix reversal generators of the symmetric and hyperoctahedral group**, *Charles Buehrle*
- **A family of Bell transformations**, *Juan B. Gil and Michael D. Weiner*
- **Permutations sorted by a finite and an infinite stack in series**, *Yoong Kuah Goh*
- **Connecting descent and peak polynomials**, *Ezgi Kantarci Oğuz*
- **Permutation packing in words of the form $\pi\pi^r$ and $\pi\pi$**, *Julia Krull, Eric Redmon, and Andrew Reimber-Berg*
- **On the distribution of statistics for pattern avoiding permutations**, *Jacob Roth*
- **A Domino tableau-based view on type B Schur-positivity**, *Ekaterina A. Vassilieva*

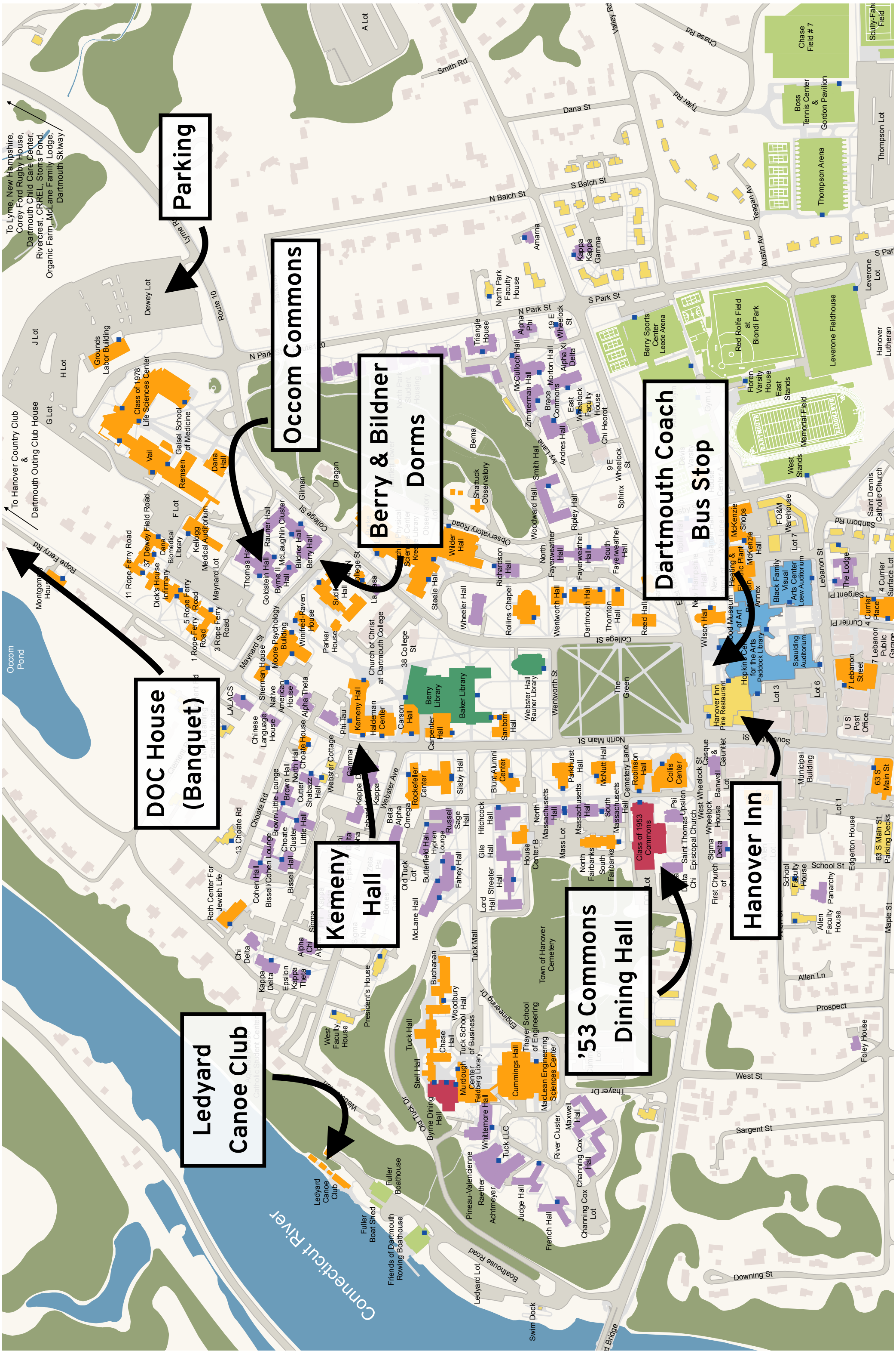
Conference Banquet

Tuesday, 6:00pm–, Dartmouth Outing Club House
The conference banquet will be held on Tuesday at the Dartmouth Outing Club House on the North edge of Occom Pond. Hors d'oeuvres and beverages will be served starting at 6pm, and we'll sit down for dinner at 7pm. Please feel free to arrive any time between 6pm and 7pm.

Conference Excursions

Wednesday afternoon There are two excursion options: hiking up Mount Cardigan and canoeing on the Connecticut River. Those who wish to hike up Mount Cardigan will hop on a bus after the last talk on Wednesday morning. A box lunch will be provided. Canoes will be reserved at the Ledyard Canoe Club (the time of the reservation will be announced early in the week). There is no extra charge for either of these options.

Campus Map



Parking
Occom Commons
Berry & Bildner Dorms
DOC House (Banquet)
Kemeny Hall
Dartmouth Coach Bus Stop
Hanover Inn
'53 Commons Dining Hall
Ledyard Canoe Club
Connecticut River
Occom Pond
The Green

2:00–8:00	*Registration in Occom Commons, Goldstein Hall*
5:30–7:30	*Welcome Reception and Registration in Occom Commons, Goldstein Hall*
8:00–	*After 8:00pm, participants staying in the dorms should call or text 1-561-635-2392 upon arrival.*

Monday, July 9

All talks are in Kemeny 008

8:00–8:50	*Registration outside of Kemeny 008*
8:50–9:00	*Conference Welcome*
9:00–9:25	**The local structure of permutations with given inversion density**, *David Bevan*
9:30–9:55	**The prolific proportion of permutations**, *Cheyne Homberger*
10:00–10:25	**Pattern avoidance in rooted forests**, *Kassie Archer*
10:30–11:00	*Refreshments*
11:00–11:25	**Local convergence for random permutations and the case of uniform ρ-avoiding permutations with $\lvert\rho\rvert = 3$**, *Jacopo Borga*
11:30–11:55	**Wilf collapse in permutation classes**, *Michael Albert*
12:00–2:00	*Lunch*
2:00–2:25	**Knots and permutations**, *Chaim Even-Zohar*
2:30–2:55	**Universal permutations**, *Michael Engen*
3:00–3:30	*Refreshments*
3:30–3:55	**A,B-minimial Stirling numbers**, *Brian Miceli*
4:00–4:25	**Enumerating permutations sortable by k passes through a pop-stack**, *Anders Claesson*
5:00–7:00	*Poster Session in Occom Commons, Goldstein Hall*

Tuesday, July 10

All talks are in Kemeny 008

9:00–9:25	**Enumerative and algebraic combinatorics of OEIS A071356**, *Chetak Hossain*
9:30–9:55	**Recognizing merge classes**, *Michal Opler*
10:00–10:25	**On the growth of merges and staircases of permutation classes**, *Jay Pantone*
10:30–11:00	*Refreshments*
11:00–11:55	**Plenary Talk: Patterns in standard young tableaux**, *Sara Billey*
12:00–2:00	*Lunch*
2:00–2:25	**Combinatorial exploration of permutation classes**, *Christian Bean*
2:30–2:55	**Automatic enumeration of grid classes**, *Unnar Erlendsson*
3:00–3:30	*Refreshments*
3:30–3:55	**On two-sided gamma-positivity for simple permutations**, *Shulamit Reches and Moriah Sigron*
4:00–4:25	**Homomorphisms on noncommutative symmetric functions and permutation enumeration**, *Yan Zhuang*
4:30–5:00	*Problem Session*
6:00–	*Banquet — Hors d'oeuvres and refreshments start at 6:00, dinner starts at 7:00. Please join us any time between 6:00 and 7:00.*

Wednesday, July 11

All talks are in Kemeny 008

The Wednesday morning session is dedicated to the memory of Jeff Remmel.

9:00–9:25	**Jeff Remmel as his students knew him**, *Jeff Liese, Brian Miceli, and Manda Riehl*
9:30–10:25	**Jeff Remmel's mathematical legacy**, *Tony Mendes*
10:30–11:00	*Refreshments*
11:00–11:25	**Classical permutation distribution in $S_n(132)$ and $S_n(123)$**, *Dun Qiu*
11:30–11:55	**Enumeration of super-strong Wilf equivalence classes of permutations**, *Ioannis Michos*
12:00–	*Free Afternoon*

Thursday, July 12

All talks are in Kemeny 008

9:00–9:25	**Pattern avoidance in Motzkin paths**, *Dan Daly*
9:30–9:55	**Rook and Wilf equivalence of integer partitions**, *Jonathan Bloom*
10:00–10:25	**Shuffle-compatibility for the exterior peak set**, *Darij Grinberg*
10:30–11:00	*Refreshments*
11:00–11:55	**Plenary Talk: Permutation classes and infinite antichains**, *Robert Brignall*
12:00–2:00	*Lunch*
2:00–2:25	**Stack sorting with increasing and decreasing stacks**, *Luca Ferrari*
2:30–2:55	**Thresholds of growth rates of sum-closed classes**, *Justin Troyka*
3:00–3:30	*Refreshments*
3:30–3:55	**On the distribution of peaks (and other statistics)**, *Lara Pudwell*
4:00–4:25	**Some $1 \times k$ generalized grid classes are context-free**, *Jakub Sliačan*

Friday, July 13

All talks are in Kemeny 008

9:00–9:25	**The undecidability of the joint embedding property for finitely-constrained hereditary graph classes**, *Samuel Braunfeld*
9:30–9:55	**Stack sorting tiers**, *Howard Skogman*
10:00–10:25	**Stack sorting r-tiers**, *Rebecca Smith*
10:30–11:00	*Refreshments*
11:00–11:25	**Unknotted cycles**, *Nathan McNew*
11:30–11:55	**Consecutive patterns in inversion sequences**, *Juan Auli*
12:00–2:00	*Lunch*
2:00–2:25	**The principal Möbius function of permutations with opposing adjacencies**, *David Marchant*
2:30–2:55	**Pattern-avoiding permutations and Dyson Brownian motion**, *Erik Slivken*
3:00–3:25	**The substitution decomposition of matchings and RNA secondary structures**, *Vince Vatter*

11:00	*Check out time for dorms (leave keys in the key drop off box)*

Where does the name Dartmouth come from?

The college is named for William Legge, the second Earl of Dartmouth, whose generous contribution of £50 helped found the college.

Why isn't it called Dartmouth University?

While the school has granted graduate and professional degrees for most of its history, Dartmouth has resisted the "university" label because of an old legal battle with the state of New Hampshire. In 1816, New Hampshire amended Dartmouth's charter to make it a public school. This lead to two schools in Hanover: the newly created Dartmouth University, which occupied the buildings, and the remnants of Dartmouth College, which continued operation in rented rooms and filed suit against the state. The College lost its suit in both local and state courts, and appealed to the United States Supreme Court. Daniel Webster, an 1801 alumnus, represented the College and delivered his famous words "It is, Sir, as I have said, a small college. And yet there are those who love it." The Supreme Court ruled in Spring 1819 in favor of the College.

What's the deal with all the Dr. Seuss stuff?

Theodor Geisel was a member of the class of 1925 and an editor of the school's humor magazine the Jack-O-Lantern. His senior year he held a party for all of the Jack-O-Lantern staff, and the party got out of hand. The College punished him by forcing him to resign from the Jack-O-Lantern. Geisel continued to write for the magazine under the pseudonym Seuss, his mother's maiden name. His graduating class voted Geisel the least likely to succeed.

Which fraternity is *Animal House* based on?

Alpha Delta, commonly referred to as AD, located on Wheelock Street across from the gym. They've recently been derecognized, so don't expect any loud parties!

Has Hanover ever been part of Vermont?

Yes, twice.

How many laps of the green equals one mile?

Roughly two and three-quarters.

Why is green the color of Dartmouth?

In the 1860s, Dartmouth decided that its sports teams needed a color. Green was chosen because "it was the only decent color that had not been taken already."

Abstracts

K Block Set Partition Patterns and Statistics

Amrita Acharyya University of Toledo

A set partition σ of $[n] = \{1, ..., n\}$ contains another set partition π if restricting σ to some $S \subseteq [n]$ and then standardizing the result gives π. Otherwise, we say σ avoids π. For all sets of patterns consisting of partitions of [3], the sizes of the avoidance classes are determined by Sagan and by Goyt. Set partitions are in bijection with restricted growth functions (RGFs) for which Wachs and White defined four fundamental statistics. Sagan, Dahlberg, Dorward, Gerhard, Grubb, Purcell, Reppuhn consider the distributions of these statistics over various avoidance classes, thus obtaining multivariate analogues of the previously cited cardinality results. They did the first in-depth study of such distributions in [1]. The analogues of their many results in [1] follows for set partitions with exactly k blocks for a specified positive integer k. These analogues are discussed in this work.

There has been an explosion of papers recently dealing with pattern containment and avoidance in various combinatorial structures. And the study of statistics on combinatorial objects has a long and venerable history. By comparison, there are relatively few papers which study a variety of statistics on a number of different avoidance classes. The focus of the present work is pattern avoidance in set partitions with k blocks combined with four important statistics defined by Wachs and White [4]. Sagan, Dahlberg, Dorward, Gerhard, Grubb, Purcell, Reppuhne, did the first comprehensive study of these statistics on avoidance classes in [1]. In particular, consider the distribution of these statistics over every class avoiding a set of partitions of $\{1, 2, 3\}$. We should note that there are other statistics on the family of all set partitions which have yielded interesting results. For example, there is a bijection between set partitions and rook placements on a triangular board. Garsia and Remmel [2] defined two statistics on rook placements giving q-analogues of the Stirling numbers of the second kind which inspired the work of Wachs and White. And set partitions avoiding certain patterns have received special attention. To illustrate, those avoiding the partition 123 are just matchings or equivalently involutions. An interesting statistic on matchings viewed in the setting of rook theory was given by Haglund and Remmel [3]. Let us start by providing the necessary definitions and setting notations according as in [1]. A set partition of a set S is a collection σ of nonempty subsets whose disjoint union is S. We write $\sigma = B_1/B_2/...B_k \vdash S$ where the subsets B_i are called blocks. When no confusion will result, we often drop the curly braces and commas in the B_i. For $[n] = \{1, ..., n\}$, we use the notation $\Pi_n = \{\sigma : \sigma \vdash [n]\}$. To define pattern avoidance in this setting, suppose $\sigma = B_1/.../B_k \in \Pi_n$ and $S \subseteq [n]$. Then σ has a corresponding sub partition σ' whose blocks are the nonempty intersections $B_i \bigcap S$. For example, if $\sigma = 14/236/5 \vdash [6]$ and $S = \{2, 4, 6\}$ then $\sigma' = 26/4$. We standardize a set partition with integral elements by replacing the smallest element by 1, the next smallest by 2, and so forth. So the standardization of σ' above is 13/2. Given two set partitions σ and π, we say that σ contains π as a pattern if there is a subpartition of σ which standardizes to π. Otherwise we say that σ avoids π. Continuing our example, we have already shown that $\sigma = 14/236/5$ contains $13/2$. But σ avoids $123/4$ because the only block of σ containing three elements also contains the largest element in σ, so there can be no larger element in a separate block. We let $\Pi_n(\pi) = \{\sigma \in \Pi_n : \sigma$ avoids $\pi\}$. In order to connect set partitions with the statistics of Wachs and White, we will have to convert them into restricted growth functions. A restricted growth function (RGF) is a sequence $w = a_1 ... a_n$ of positive integers subject to the restrictions

1. $a_1 = 1$.
2. For $i \geq 2$ we have $a_i \leq 1 + max\{a_1, ..., a_{i-1}\}$

The number of elements in w is called its length and let $\Pi_{n,k}$ be the set of all words in Π_n with exactly k blocks. We let $\Pi_{n,k}(\pi)$ be the set of all words in Π_n with exactly k blocks that avoids π, $R_n = \{w : w$ is an RGF of length $n\}$. Let, $R_{n,k} = \{w : w$ is an RGF of length n with exactly k blocks $\}$, that is $R_{n,k} = \{w : w$ is an RGF of length n with maximal letter $k\}$. There is a simple bijection $\Pi_n \mapsto R_n$. We say $\sigma = B_1/.../B_k \in \Pi_n$ is in standard form if min $B_1 <$ ... $<$ min B_k. Note that this forces min $B_1 = 1$. We henceforth assume all partitions in Π_n are written in standard form. Associate with σ the word $w(\sigma) = a_1 ... a_n$ where $a_i = j$ if and only if $i \in B_j$. Using the example from the previous paragraph $w(\sigma) = 122132$. It is easy to see that $w(\sigma)$ is a restricted growth function and that the map $\sigma \mapsto w(\sigma)$ is the desired bijection. Note that the restriction of the above bijection is a bijection from $\Pi_{n,k} \mapsto R_{n,k}$. It will be useful to have a notation for the RGFs of partitions avoiding a given pattern π, namely $R_n(\pi) = \{w(\sigma) : \sigma \in \Pi_n(\pi)\}$. One can express the notion of partition pattern avoidance directly in the language of restricted growth functions. The canonization of a sequence $v = b_1 ... b_k$ of integers is obtained by replacing all copies of the first element of v by 1, all copies of the second different element to appear in v by 2, and so on. For example $v = 55533522312$ canonizes to 11122133243. It follows easily from the definition that the canonization of v is always an RGF. Sagan described the set partitions in $\Pi_n(\pi)$ for each $\pi \in \Pi_3$. Define the initial run of an RGF w to be the longest prefix of the form $12...m$. Also, we will use the notation a^l to indicate a string of l consecutive copies of the letter a in a word. Finally, say that w is layered if $w = 1^{n_1} 2^{n_2} ... m^{n_m}$ for positive integers $n_1, n_2, ..., n_m$. It will be further useful to have a notation for the RGFs of partitions with exactly k blocks avoiding a given pattern π, namely $R_{n,k}(\pi) = \{w(\sigma) : \sigma \in \Pi_{n,k}(\pi)\}$.

We started developing the analogues from the analogue of **Theorem 1.2** in [1].

References

[1] S. Dahlberg, R. Dorward, J. Gerhard, T. Grubb, C. Purcell, L. Reppuhn, and B. E. Sagan. Set partition patterns and statistics. *Discrete Math.*, 339(1):1–16, 2016.

[2] A. M. Garsia and J. B. Remmel. *Q*-counting rook configurations and a formula of Frobenius. *J. Combin. Theory Ser. A*, 41(2):246–275, 1986.

[3] J. Haglund and J. B. Remmel. Rook theory for perfect matchings. *Adv. in Appl. Math.*, 27(2-3):438–481, 2001.

[4] M. Wachs and D. White. p, q-Stirling numbers and set partition statistics. *J. Combin. Theory Ser. A*, 56(1):27–46, 1991.

Wilf Collapse in Permutation Classes

Michael Albert University of Otago

This talk is based on joint work with Vít Jelínek and Michal Opler

Let $\mathcal{C}$ be a permutation class and consider the equivalence relation $\sim$ defined on $\mathcal{C}$ by $\pi \sim \tau$ whenever the two permutation classes $\mathcal{C}_\pi := \mathcal{C} \cap \mathrm{Av}(\pi)$ and $\mathcal{C}_\tau := \mathcal{C} \cap \mathrm{Av}(\tau)$ are Wilf-equivalent, i.e., have the same generating function.

Clearly $\pi \sim \tau$ implies $|\pi| = |\tau|$ so for each positive integer n it makes sense to define $w_n = |\mathcal{C}_n/{\sim}|$ (where $\mathcal{C}_n$ is the set of permutations in $\mathcal{C}$ of size n) and $c_n = |\mathcal{C}_n|$ and to ask:

Question 1. *Aside from the trivial observation that $w_n \leq c_n$ what can be said about the relationship between w_n and c_n?*

In previous work Mathilde Bouvel and I had observed that for $\mathcal{C} = \mathrm{Av}(312)$, $w_n = o(2.5^n)$ while of course $c_n^{1/n} \to 4$ so in particular $w_n < r^n c_n$ for some constant $r < 1$ and all sufficiently large n. We call this phenomenon an *exponential Wilf-collapse*. More formally:

Definition 2. The class $\mathcal{C}$ exhibits a *Wilf-collapse* if $w_n = o(c_n)$, and an *exponential Wilf-collapse* if there is some constant $r < 1$ such that $w_n = o(r^n c_n)$.

Given the attention that has been focussed on particular instances of Wilf-equivalence one might expect that Wilf-collapses, let alone exponential Wilf-collapses are rare. It seems though that this is not the case. Aside from many specific examples (such as $\mathrm{Av}(312)$ noted above) we can prove the following:

Theorem 3. *Let $\mathcal{C}$ be any sum-closed class having only finitely many sum-indecomposable permutations. Then $\mathcal{C}$ has an exponential Wilf-collapse unless $\mathcal{C}$ is the class of increasing permutations.*

Furthermore,

Theorem 4. *Let $\mathcal{C}$ be any class having only finitely many sum-indecomposable permutations. Then $\mathcal{C}$ has a Wilf-collapse unless (c_n) is a bounded sequence.*

While these results are quite specific they seem to fit into a broader framework that we hope can be used to show that (exponential) Wilf-collapse is the 'expected' behaviour in most permutation classes.

Rooted Forests That Avoid Sets of Permutations

Kassie Archer The University of Texas at Tyler

This talk is based on joint work with Katie Anders

Our objects of study will be unordered (i.e. non-planar) rooted forests. Properties of these forests have previously been studied extensively. In particular, there are many interesting results regarding statistics on these trees and forests, such as descents (both vertex-descents and edge-descents) [6, 7, 4], major index [11], inversions [12, 11], leaves [6], hook length [3, 5], leaders (analogs of right-to-left minima) [7, 13, 8], and many others. Additionally, increasing trees and forests, which avoid the pattern 21 in our context, have been widely studied and are useful combinatorial objects, as in [10, 9, 1], and are easily enumerated (see [15]). Alternating trees, which avoid the consecutive patterns 321 and 123 in our context, have also been enumerated in [2].

We study unordered rooted forests which avoid sets of permutations. In particular, we show that there are two forest-Wilf classes for patterns of length three and we provide enumerations for the sets of forests on $[n]$ that avoid given sets of permutations of lengths 3 and 4 by constructing bijections with certain types of set-partitions of $[n]$ or lists with special properties. The table in Figure 1 summarizes the results found so far.

In the table in Figure 1, $c(n,k)$, $S(n,k)$, and $B(n)$ denote the unsigned Stirling numbers of the first kind, the Stirling numbers of the second kind, and the Bell numbers, respectively. The unsigned Stirling numbers of the first kind, denoted by $c(n,k)$ (or $\left[{n \atop k}\right]$), enumerate the permutations in $\mathcal{S}_n$ which can be decomposed into k disjoint cycles. The Stirling numbers of the second kind, denoted by $S(n,k)$, is the number of ways to partition the set $[n]$ into k nonempty subsets. The Bell numbers, denoted by $B(n)$ is the number of ways to partition the set $[n]$ into subsets, i.e. it is the sum of the Stirling numbers of the second kind for a fixed n.

Rooted forests

Let F_n denote the set of unordered rooted labeled forests on $[n]$. We draw forests as rooted trees with an unlabeled root as in Figure 2.

We say that a labeled rooted forest on $[n]$ avoids the pattern $\sigma \in \mathcal{S}_k$ if along each path from the root to a vertex, the sequence of labels $i_1 i_2 i_3 \dots i_m$ do not contain a subsequence that is in the same relative order as $\sigma_1\sigma_2\dots\sigma_k$. Let $F_n(\sigma)$ denote the set of forests on $[n]$ that avoid σ and let $f_n(\sigma)$ be the number of such forests. Notice that $F_n(21)$ is the set of increasing forests on $[n]$ and $F_n(12)$ is the set of decreasing forests on $[n]$.

It is well known that the number of increasing forests on $[n]$ is $f_n(21) = n!$. (Similarly,

Forests	Enumeration
$F_n(213, 312)$	$\sum_{k=1}^{n} k!\, c(n,k)$
$F_n(231, 132)$	
$F_n(213, 312, 123)$	$\sum_{k=1}^{n} \mathcal{B}(n)\, c(n,k)$
$F_n(231, 132, 321)$	
$F_n(213, 312, 321)$	$\sum_{k=1}^{n} k!\, S(n,k)$
$F_n(231, 132, 123)$	
$F_n(312, 213, 132)$	$\sum_{k=1}^{n} \frac{n!}{k!}\binom{n-1}{k-1}$
$F_n(132, 231, 312)$	
$F_n(321, 132, 213)$	
$F_n(123, 312, 231)$	
$F_n(213, 312, 231)$	recurrence
$F_n(231, 132, 213)$	
$F_n(321, 2143, 3142)$	$n! + \sum \frac{n!}{2^\ell}\binom{n-k-1}{\ell-1}\binom{k}{\ell}$
$F_n(123, 3412, 2413)$	

Figure 1: For each forest class listed, the sets of forests that avoid the certain sets of permutations are enumerated. The sum associated to $F_n(321, 2143, 3142)$ and $F_n(123, 3412, 2413)$ occurs over all $1 \le \ell \le k \le n$ so that $\ell + k \le n$.

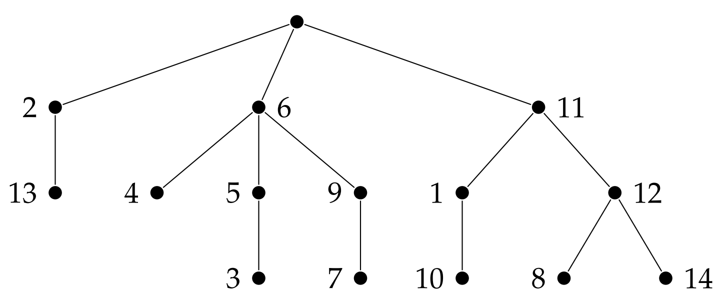


Figure 2: An example of a rooted forest on [14]. There are three trees in this forest with roots labeled 2, 6, and 11.

the number of decreasing forests is $f_n(12) = n!$.) Perhaps the easiest way to see this is via induction. Any increasing forest on $[n]$ must have the label n assigned to a leaf. On a forest of $n-1$ elements, there are n places one can add such a leaf (as a child of the $n-1$ vertices or as a root). Thus $f_n(21) = n \cdot f_{n-1}(21)$. Since there is one forest of size one, there must be $n!$ increasing forests. A natural bijection, which we call φ, between permutations on an ordered set A and increasing forests on A can be described in the following way (as in [15]): Given $\pi \in \mathcal{S}_n$, the forest $\varphi(\pi)$ is obtained by letting all left-right minima be roots of the trees. For the remaining vertices, let i be the child of the rightmost element j of π that precedes i and is less than i. Several of the proofs make use of this bijection.

In Figure 3, we see an example of this bijection. If $\pi = 3, 6, 8, 4, 1, 10, 2, 9, 7, 5$, then the left-to-right minima 3 and 1 will be roots of trees in the increasing forest. Since 3 is the rightmost element to the left of 6 that is smaller than it, 6 will be a child of 3. Similarly, 8 will be a child of 6. Since 3 is the rightmost element to the left of 4 that is smaller than it, 4 will be a child of 3, and so on. The inverse of this bijection is to traverse the forest clockwise starting at the root (after ordering the children of each vertex least to greatest from left to right as in Figure 3), reading off each label as it is reached.

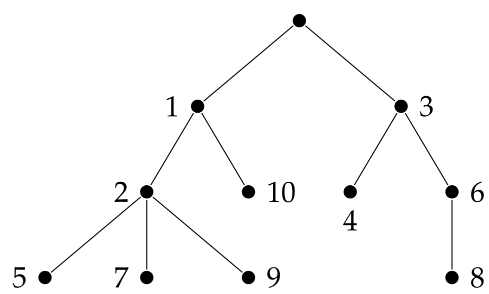


Figure 3: The map φ sends a permutation of $[n]$ to an increasing forest on $[n]$. The forest pictured here is $\varphi(3, 6, 8, 4, 1, 10, 2, 9, 7, 5)$.

Forest-Wilf-equivalence for patterns of size 3

An analog of Wilf-equivalence can be defined for rooted forests as follows. The pattern σ is *forest-Wilf-equivalent* to the pattern τ if $f_n(\sigma) = f_n(\tau)$. A set of patterns $S = \{\sigma_1, \ldots, \sigma_k\}$ is *forest-Wilf-equivalent* to a set of patterns $T = \{\tau_1, \ldots, \tau_\ell\}$ if $f_n(\sigma_1, \ldots, \sigma_k) = f_n(\tau_1, \ldots, \tau_\ell)$. The trivial symmetry of complementation of permutations can be adapted to this setting. There is no clear analog of reverse or inverse for forests that preserves forest-Wilf-equivalence.

In this section, we determine that there are two forest-Wilf-equivalence classes for patterns of length three, in contrast to the case of permutations where there is only one Wilf-equivalence class. By direct computation, it has been determined that $f_n(321) = f_n(231)$ for $n = 1, 2, 3, 4$. However, $f_5(321) = 918$, while $f_5(231) = 917$. However, we can construct a bijection between forests avoiding 312 and those avoiding 321. The process is similar to that in the proof of Simion and Schmidt from [14].

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Consecutive Patterns in Inversion Sequences

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This talk is based on joint work with Sergi Elizalde

A very common encoding of a permutation π is its inversion sequence $\Theta(\pi) = e = e_1e_2\cdots e_n$, with $e_i = \left|\{j : j < i \text{ and } \pi_j > \pi_i\}\right|$. The map $\Theta : S_n \to \mathbf{I}_n$ is one of several bijections between the set permutations of $[n]$, S_n, and the set of inversion sequences of length n, $\mathbf{I}_n$. This correspondence led Corteel, Martinez, Savage and Weselcouch [4] and Mansour and Shattuck [7] to begin a systematic study of classical pattern avoidance in inversion sequences.

The reduction of an inversion sequence e is defined to be the word obtained by replacing the ith smallest entry of e with $i-1$. We say e contains the (classical) pattern $p = p_1p_2\cdots p_k \in \{0,1,\ldots,k-1\}^k$ if there exist a subsequence $e_{i_1}e_{i_2}\cdots e_{i_k}$ of e with reduction p. Otherwise, we say e avoids p.

Example 1. The inversion sequence $e = 0021213$ avoids the pattern 210, but it contains the pattern 110. Indeed, $e_3e_5e_6 = 221$ has reduction 110.

Note that unlike permutation patterns, an inversion sequence pattern may have repeated entries.

Corteel et al. [4] and Mansour and Shattuck [7] provide enumerative results for the sets $\mathbf{I}_n(p)$, consisting of inversion sequences of length n avoiding the pattern p, for patterns p of length 3. Their results connect classical patterns in inversion sequences to many integer sequences, such as Fibonacci numbers, Bell numbers, Schröder numbers and Euler up/down numbers.

We consider consecutive patterns in inversion sequences in an analogous manner to consecutive permutation patterns, see [6]. We say an inversion sequence e contains the consecutive pattern $p = \underline{p_1p_2\cdots p_k}$ if there is a consecutive subsequence $e_ie_{i+1}\cdots e_{i+k-1}$ of e with reduction p. Otherwise, we say e avoids p. We underline the entries of p to distinguish it from the classical case.

Example 2. The inversion sequence $e = 002031$ contains the classical pattern 120 because $e_3e_5e_6 = 231$ has reduction 120. However, e avoids the consecutive pattern $\underline{120}$.

We initiate a systematic study of consecutive patterns in inversion sequences in an analogous manner to that of Corteel et al. [4] and Mansour and Shattuck [7]. Namely, we give enumerative results for the set $\mathbf{I}_n(p)$ consiting of the inversion sequences avoiding the consecutive pattern p, for p of length 3. Our results connect consecutive patterns in inversion sequences to a number of well-known integer sequences, including Catalan numbers, Fibonacci numbers, central polynomial numbers and little Schröeder numbers.

Additionally, we classify patterns of length 3 and 4 according to the following.

Definition 3. We say two consecutive patterns p and p' are *Wilf equivalent*, which we denote by $p \sim p'$, if $|\mathbf{I}_n(p)| = |\mathbf{I}_n(p')|$ for all n.

On the other hand, p and p' are *strongly Wilf equivalent*, denoted $p \overset{s}{\sim} p'$, if for each n and k, the number of inversion sequences in $\mathbf{I}_n$ containing k occurrences of p is the same as for p'.

If the above condition holds for any set of prescribed positions for the k occurrences, then we say p and p' are *super strongly Wilf equivalent* and we denote this by $p \overset{ss}{\sim} p'$.

These definitions are analogous to existing ones for consecutive permutation patterns, see [5]. Henceforth, unless otherwise stated, whenever we say two patterns are equivalent, we mean with respect to inversion sequences. Furthermore, we refer to an equivalence of any one of the three types introduced as a generalized Wilf equivalence.

Our results for consecutive patterns of length 3 are, for the most part, recurrences, using the refinement $\mathbf{I}_{n,k} = \{e \in \mathbf{I}_n : e_n = k\}$ of $\mathbf{I}_n$, which allows us to define

$$\mathbf{I}_{n,k}(p) = \mathbf{I}_{n,k} \cap \mathbf{I}_n(p)$$

for a given pattern p. Note that $\mathbf{I}_{n,k}(p) = \emptyset$ for $k \geq n$ and $\mathbf{I}_n(p) = \cup_{k \geq 0} \mathbf{I}_{n,k}(p)$ for any p.

Example 4. For the consecutive pattern $p = \underline{001}$, we have $|\mathbf{I}_3(\underline{001})| = 4$, as $\mathbf{I}_{3,0}(\underline{001})$= {000, 010}, $\mathbf{I}_{3,1}(\underline{001})$={011} and $\mathbf{I}_{3,2}(\underline{001})$={012}.

A summary of our results for consecutive patterns of length 3 is given in Table 1, for patterns with no repeated letters, and in Table 2, for patterns with repeated letters. In particular, we indicate which sequences match existing sequences in the On-Line Encyclopedia of Integer Sequences (OEIS) [10].

Pattern p	in the OEIS?	Observations about $\lvert\mathbf{I}_n(p)\rvert$
$\underline{012}$	A049774	Counts $\lvert S_n(\underline{321})\rvert$
$\underline{021}$	A071075	Counts $\lvert S_n(\underline{132}4)\rvert$
$\underline{102}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - \sum_{j \geq 1} j \cdot \left\lvert\mathbf{I}_{n-2,j}(p)\right\rvert$
$\underline{120}$	A200404	Counts $\lvert S_n(\underline{143}2)\rvert$
$\underline{201}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - k\sum_{j>k} \left\lvert\mathbf{I}_{n-2,j}(p)\right\rvert$
$\underline{210}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - \sum_{m>k}\sum_{j>m}\sum_{i \leq j} \lvert\mathbf{I}_{n-3,i}(p)\rvert$

Table 1: Consecutive patterns of length 3 with no repeated letters.

Here $\underline{132}4$ is a vincular or generalized permutation pattern, see [1].

Remark 5. Table 2 shows that $\underline{100} \overset{ss}{\sim} \underline{110}$. This is the only pair of consecutive patterns of length 3 that are super strongly Wilf equivalent. In fact, no other consecutive patterns of length 3 are even Wilf equivalent.

Pattern p	in the OEIS?	Observations about $\lvert\mathbf{I}_n(p)\rvert$
$\underline{000}$	A052169	Counted by $\frac{(n+1)!-d_{n+1}}{n}$, where d_n is the number of derangements of [n]
$\underline{001}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - \sum_{j<k}\lvert\mathbf{I}_{n-2,j}(p)\rvert$
$\underline{010}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - (n-2-k)\lvert\mathbf{I}_{n-2,k}(p)\rvert$
$\underline{011}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - \sum_{j<k}\lvert\mathbf{I}_{n-2,j}(p)\rvert$ if $k \neq n-1$, and $\lvert\mathbf{I}_{n,n-1}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert$
$\underline{100} \overset{ss}{\sim} \underline{110}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - \sum_{j>k}\lvert\mathbf{I}_{n-2,j}(p)\rvert$
$\underline{101}$	New	$\lvert\mathbf{I}_{n,k}(p)\rvert = \lvert\mathbf{I}_{n-1}(p)\rvert - k\lvert\mathbf{I}_{n-2,k}(p)\rvert$

Table 2: Consecutive patterns of length 3 with repeated letters.

The following theorem describes the generalized Wilf equivalence classes for consecutive patterns of length 4.

Theorem 6. *A complete list of the generalized Wilf equivalences between consecutive patterns of length* 4 *is as follows:*

(i) $\underline{2013} \overset{ss}{\sim} \underline{2103}$

(ii) $\underline{3012} \overset{ss}{\sim} \underline{3102}$

(iii) $\underline{0100} \overset{ss}{\sim} \underline{0110}$

(iv) $\underline{1000} \overset{ss}{\sim} \underline{1110}$

(v) $\underline{1101} \overset{ss}{\sim} \underline{1001} \overset{ss}{\sim} \underline{1011}$

(vi) $\underline{0021} \overset{ss}{\sim} \underline{0121}$

(vii) $\underline{1002} \overset{ss}{\sim} \underline{1102} \overset{ss}{\sim} \underline{1012}$

(viii) $\underline{2001} \overset{ss}{\sim} \underline{2101} \overset{ss}{\sim} \underline{2011} \overset{ss}{\sim} \underline{2201}$

(ix) $\underline{1200} \overset{ss}{\sim} \underline{1210} \overset{ss}{\sim} \underline{1220}$

(x) $\underline{2100} \overset{ss}{\sim} \underline{2210}$

(xi) $\underline{0211} \overset{ss}{\sim} \underline{0221}$

(xii) $\underline{2012} \overset{ss}{\sim} \underline{2102}$

This fact, and Remark 5, led us to speculate the following.

Conjecture 7. *Two consecutive inversion sequence patterns of length k are strongly Wilf equivalent if and only if they are Wilf equivalent.*

This conjecture is completely analogous to one by Nakamura [9, Conjecture 5.6] for consecutive permutation patterns, which remains open.

A stronger version of Conjecture 7 may hold, as it is possible that all three types of generalized Wilf equivalence classes coincide for all k. We show this is true for consecutive patterns of length $k \leq 4$.

Continuing the systematic study started by Corteel et al. [4], Martinez and Savage [8] reframe the notion of a length-3 pattern to consider a triple of binary relations, as opposed to a word of length 3. Given a fixed triple of binary relations (R_1, R_2, R_3), where $R_i \in \{\leq, \geq, <, >, =, \neq, -\}$, they study the set $\mathbf{I}_n(R_1, R_2, R_3)$ consisting of those $e \in \mathbf{I}_n$ with no subindices $i < j < k$ such that $e_i R_1 e_j$, $e_j R_2 e_k$ and $e_i R_3 e_k$. The relation "$-$" on a set S means all of $S \times S$. That is, $x - y$ for all $x, y \in S$

Example 8. $\mathbf{I}_n(\geq, \leq, \leq)$ is the set of inversion sequences $e \in \mathbf{I}_n$ with no $i < j < k$ such that $e_i \geq e_j$, $e_j \leq e_k$ and $e_i \leq e_k$. That is, the set of inversion sequences in $\mathbf{I}_n$ avoiding all the words in the set $\{000, 001, 101, 102\}$. Then $\mathbf{I}_n(\geq, \leq, -)$ would denote the set of inversion sequences avoiding all the words in the set $\{000, 001, 100, 101, 102, 201\}$.

We study the consecutive analogues of the sets $\mathbf{I}_n(R_1, R_2, -)$. Namely, the sets $\mathbf{I}_n(R_1, R_2)_c$, with $R_i \in \{\leq, \geq, <, >, =, \neq\}$, consisting of inversion sequences $e \in \mathbf{I}_n$ in which the relations R_1 and R_2 do not occur consecutively. That is, there is no subindex i such that $e_i R_1 e_{i+1}$ and $e_{i+1} R_2 e_{i+2}$.

Example 9. An inversion sequence e avoids the consecutive pattern $(\geq, =)_c$ if there is no subindex i such that $e_i \geq e_{i+1} = e_{i+2}$. Thus, $\mathbf{I}_n(\geq, =)_c = \mathbf{I}_n(\underline{000}) \cap \mathbf{I}_n(\underline{100})$.

As in the case of consecutive patterns understood as words, we define a notion of generalized Wilf equivalence for tuples of consecutive relations.

Definition 10. Two patterns $(R_1, R_2)_c$ and $(R_1', R_2')_c$ of tuples of consecutive relations are *Wilf equivalent*, denoted by $(R_1, R_2)_c \sim (R_1', R_2')_c$, if $|\mathbf{I}_n(R_1, R_2)_c| = |\mathbf{I}_n(R_1', R_2')_c|$ for all n.

We say $(R_1, R_2)_c$ and $(R_1', R_2')_c$ are *strongly Wilf equivalent*, denoted by $(R_1, R_2)_c \overset{s}{\sim} (R_1', R_2')_c$, if for each n and k, the number of inversion sequences in $\mathbf{I}_n$ containing k occurrences of $(R_1, R_2)_c$ is the same as for $(R_1', R_2')_c$.

If the above condition holds for any set of prescribed positions for the k occurrences, then we say $(R_1, R_2)_c$ and $(R_1', R_2')_c$ are *super strongly Wilf equivalent* and we denote this by $(R_1, R_2)_c \overset{ss}{\sim} (R_1', R_2')_c$.

We refer to an equivalence of any one of these three types as a generalized Wilf equivalence.

There are 36 patterns $(R_1, R_2)_c$ of tuples of consecutive relations. A classification of these patterns into generalized Wilf equivalence classes is provided by the next result.

Theorem 11. *A complete list of the generalized Wilf equivalences between consecutive patterns of inequalities of length* 3 *is as follows:*

(i) $(\geq, \geq)_c \overset{ss}{\sim} (<, <)_c$

(ii) $(\geq, <)_c \overset{ss}{\sim} (<, \geq)_c \sim (\neq, \geq)_c$

(iii) $(\geq, >)_c \overset{ss}{\sim} (>, \geq)_c$

(iv) $(\geq, =)_c \overset{ss}{\sim} (=, \geq)_c$

(v) $(>, =)_c \overset{ss}{\sim} (=, >)_c$

Although the patterns $(<, \geq)_c$ and $(\neq, \geq)_c$ are Wilf equivalent, they are not strongly Wilf equivalent. This shows that strong and Wilf equivalence classes do not coincide. Hence, an analogous result to Conjecture 7 for patterns of tuples of consecutive relations is false.

As a consequence of Theorem 11 *(iii)*, we deduce the following.

Corollary 12. *The permutation patterns* $\underline{124}3$ *and* $\underline{421}3$ *are Wilf equivalent.*

This result was originally conjectured by Baxter and Pudwell [2, Conjecture 17] and then proved by Baxter and Shattuck [3, Corollary 11]. We present a simpler and more direct proof of this fact.

In addition to classifying patterns of tuples of consecutive relations into generalized Wilf equivalence classes, we prove that for many patterns $(R_1, R_2)_c$ the sequence $|\mathbf{I}(R_1, R_2)_c|$ has an alternative combinatorial interpretation. In fact, in several cases they coincide with the avoidance sequence of a permutation pattern. These results are summarized in Table 3.

Pattern p	$\|\mathbf{I}_n(p)\|$ in the OEIS as	OEIS description
$(\leq, \geq)_c$	A000027	n
$(\leq, >)_c$	A000108	C_n: Catalan numbers
$(\leq, \neq)_c$	A040000	$1, 2, 2, \ldots$ (constant 2 for $n > 1$)
$(\geq, \leq)_c$	A000045	F_{n+2}: $(n+2)$th Fibonacci number
$(\geq, \geq)_c \overset{ss}{\sim} (<, <)_c$	A049774	$\|S_n(\underline{321})\|$
$(\geq, <)_c \overset{ss}{\sim} (<, \geq)_c \overset{s}{\sim} (\neq, \geq)_c$	A000079	2^{n-1}
$(\geq, >)_c \overset{ss}{\sim} (>, \geq)_c$	A200403	$\|S_n(\underline{124}3)\|$
$(\geq, \neq)_c$	A000124	Central polygonal numbers (lazy caterer's sequence)
$(>, \leq)_c$	A071356	Motzkin paths of length n with up and level steps coming in two colors
$(=, =)_c$	A052169	$\frac{(n+1)!-d_{n+1}}{n}$
$(\neq, \leq)_c$	A000071	$F_{n+2} - 1$, where F_{n+2} is the $(n+2)$th Fibonacci number
$(\neq, =)_c$	A000522	Number of 01−avoiding rook monoids
$(\neq, \neq)_c$	A000085	Number of involutions of $[n]$

Table 3: Patterns p of tuples of consecutive relations for which $|\mathbf{I}_n(p)|$ has an alternative combinatorial interpretation.

We show that $\mathbf{I}_n(>, \leq)_c = \mathbf{I}_n(>, \leq, -)$ for all n. As a consequence, we obtain the following result, which was conjectured by Martinez and Savage [8, Section 2.19].

Corollary 13. *The sequence $|\mathbf{I}_n(>, \leq, -)|$ has ordinary generating function*

$$\frac{1 - 2x - \sqrt{1 - 4x - 4x^2}}{4x^2}.$$

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Combinatorial Exploration of Permutation Classes

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This talk is based on joint work with Michael Albert, Anders Claesson, Tomas Ken Magnusson, Jay Pantone, and Henning Úlfarsson

In recent years there has been increasing interest in automated provers and proof checkers for mathematical statements. Most of the tools developed have been general purpose and therefore unable to leverage domain-specific knowledge. The purpose of this work is to develop an algorithm for proving theorems in the area of enumerative combinatorics, in particular, permutation patterns.

The design of our tool is such that it can be easily adapted to other areas of mathematics by including suitable domain-specific knowledge. This approach would be particularly viable for problems of determining or verifying regularity in collections of discrete structures.

The CombSpecSearcher algorithm

The goal of enumerative combinatorics is to enumerate a given combinatorial object. One method is to find a *combinatorial specification* as defined in [6]. We will review some of these definitions briefly. A *combinatorial class* is a set, $\mathcal{C}$, with a size function $\mathcal{C} \mapsto \mathbb{N}_0$ such that the preimage for all integers is finite. Any element $c \in \mathcal{C}$ is called a *combinatorial object* and the size of c is denoted $|c|$. For a combinatorial class, we let $\mathcal{C}_n$ be the set of objects in $\mathcal{C}$ of size n. The goal is to determine the sequence $(|\mathcal{C}_n|)_{n\geq 0}$.

A *(combinatorial) rule* is a tuple $(\mathcal{C}, \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}, \circ)$, where $\mathcal{C}$ and the $\mathcal{C}_i$ are combinatorial classes and $\circ$ is an admissable constructor, if $\mathcal{C} \cong \mathcal{C}_1 \circ \mathcal{C}_2 \circ \cdots \circ \mathcal{C}_k$, that is, there is a length preserving bijection between the $\mathcal{C}$ and $\mathcal{C}_1 \circ \mathcal{C}_2 \circ \cdots \circ \mathcal{C}_k$. The admissable constructors we have here are disjoint union and Cartesian product. If the enumeration of a combinatorial class is known then the combinatorial rule $(\mathcal{C}, \emptyset, \circ)$ is allowed.

A *combinatorial specification* for a tuple $\check{\mathcal{C}} = (\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k)$ of non-empty combinatorial classes is a set of k rules using only combinatorial classes in $\check{\mathcal{C}}$ and each $\mathcal{C}_i$ appears on the left of a rule once. We say it is a combinatorial specification for $\mathcal{C}_1$ also.

There are two main stages to *combinatorial exploration*. The first is the expansion stage, that is to apply *strategies* to the combinatorial classes of interest, and create a set of *rules* which explain how combinatorial classes are related. The second stage is to check if there exists a structural description for the original combinatorial class of interest within this set of rules. This procedure has been automated, and we call our algorithm CombSpecSearcher.

The TileScope algorithm

We now focus our attention on enumerating the number of length n permutations in a permutation class. There have been some systematic approaches towards getting this enumeration, by which we mean a polynomial time algorithm in n to compute $|\mathrm{Av}_n(\Pi)|$ as in [11]. The four widely applicable techniques meeting this criteria are generating trees [10], enumeration schemes [7, 12], substitution decomposition [1, 4] and the insertion encoding [3, 9]. An overview of these four algorithms can be found here [7].

The work here builds upon the structures found by the Struct algorithm [5], which conjectured structural descriptions of permutation classes using something similar to generalized grid classes as were first defined by [8]. We use a refinement of this that we call *gridded permutations* and *tilings*.

We are going to build on the general framework provided by the CombSpecSearcher algorithm, and describe strategies which can be used to find combinatorial specifications for permutation classes. We will describe multiple strategies, and in essence, describe multiple different algorithms, but collectively we will call these the TileScope algorithm. For example, we can translate the methods of other existing automatic methods into strategies for TileScope.

Theorem 1. *The methods used by the regular insertion encoding and the original enumeration schemes as in [12] can be translated into strategies using gridded permutations and tilings.*

Our implementations of these algorithms are superior as the original algorithms rely on results that require the generation of permutations which is expensive. Our algorithm does not generate permutations.

Successes

As training ground we have been testing the TileScope algorithm on bases consisting of length 4 patterns. Given that it can be seen from the basis whether or not a permutation class is regular insertion encodable [3], and in light of Theorem 1, we consider those which are not. There exists a combinatorial specification using the TileScope for all bases with 8 or more patterns, using the strategies discussed in a later section: *requirement insertion*, *point placement*, *factors*, and *row and column separation*. In Table 1 we show the statistics for bases which TileScope has found a combinatorial specification. There are 130 bases left in this set and we have methods and ideas which can enumerate some beyond what we will discuss in this abstract. In some cases the combinatorial specification the algorithm finds for a class $\mathrm{Av}(\Pi)$ extends to any class $\mathrm{Av}(\Pi \cup \mathrm{T})$, where T is a finite set of patterns.

In Figure 1, there is an example of how to translate the classic proof for 231-avoiders into the language of gridded permutations and tilings. It uses the four strategies that were used to find the successes in Table 1. The details of what this represents are

Number of length 4 patterns	Not regular insertion encodable	Enumerated Structural description by TileScope
8	337	337
7	547	545
6	659	647
5	578	548
4	363	320
3	153	103
2	43	14
1	7	0

Table 1: The successes for bases consisting of length 4 patterns.

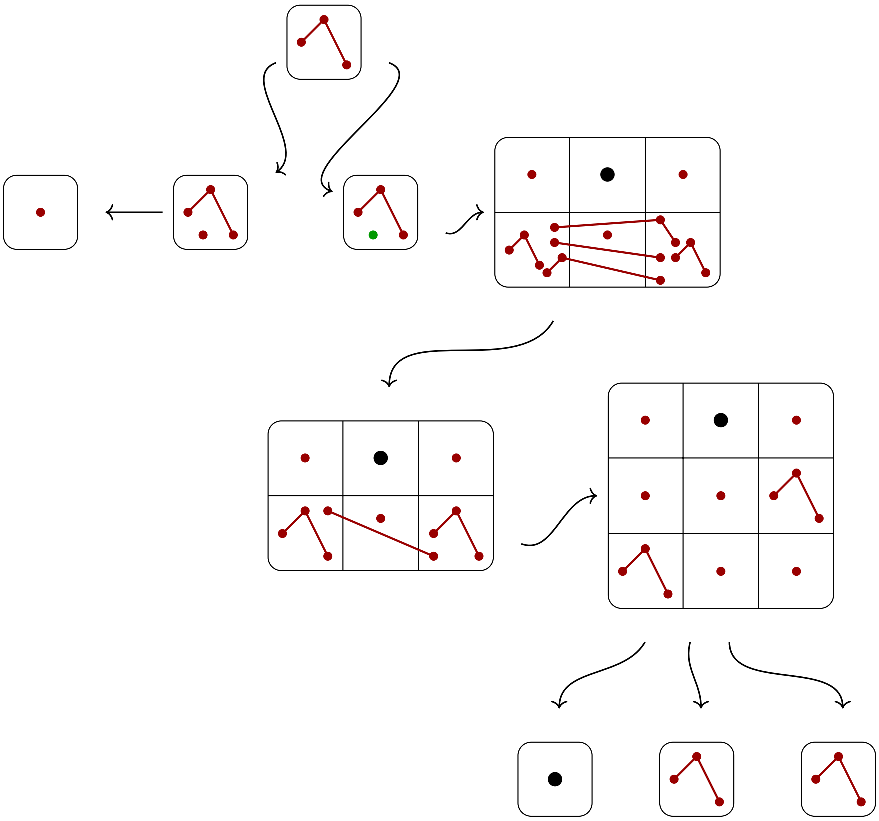

Figure 1: A pictorial representation for the combinatorial specification of Av (231).

discussed in the following sections. Roughly speaking, the red permutations you see are those which must be avoided. The green permutations are those which must be contained and the black point represents a point in the permutation, in this case, the value n in the permutation.

Gridded permutations

In this section, we will describe the combinatorial classes that we use for searching for combinatorial specifications.

We tend to think of permutations with a geometric mindset. In order to do this, we borrow some definitions from Section 2 of [2] altering them slightly to suit our thinking. The *gridded position* of a point (x, y) in $\mathbb{R}^2$ is defined as the integer point (a, b) such that $(x, y) \in [a, a+1) \times [b, b+1)$.

We say that a figure $\mathcal{F} \subseteq \mathbb{R}^2$ is *gridded involved* in the figure $\mathcal{G}$, denoted $\mathcal{F} \preceq \mathcal{G}$ if there are subsets $A, B \subseteq \mathbb{R}$ and increasing injections $\phi_x : A \mapsto \mathbb{R}$ and $\phi_y : B \mapsto \mathbb{R}$ such that

$$\mathcal{F} \subseteq A \times B \text{ and } \phi(\mathcal{F}) = \{(\phi_x(a), \phi_y(b)) : (a, b) \in \mathcal{F}\} \subseteq \mathcal{G}$$

and ϕ preserves the gridded position of a point (x, y) in $\mathbb{R}^2$. If you ignore the gridded position you recover the definition of involvement as in [2].

As in [2] this relation forms a preorder on the collection of all figures. If $\mathcal{F} \preceq \mathcal{G}$ and $\mathcal{G} \preceq \mathcal{F}$ we say they are gridded equivalent. (This means two figures are equivalent if and only if one can be transformed to the other by stretching and shrinking the axes without ever crossing the integer lattice).

We say a figure is *independent* if no two points lie on the same horizontal or vertical line. The set of all *gridded permutations*, $\mathcal{G}$, is then defined as the set of all equivalence classes of finite independent figures with respect to gridded equivalence. We define containment of gridded permutations with respect to the involvement relation, and we refer to σ as a gridded pattern when $\sigma \preceq \pi$ for some gridded permutation π. Define the *length* of a gridded permutation as the number of points. The standardization of the set of sorted points is called the *underlying permutation*.

For convenience to represent an equivalence class we write the underlying permutation where we put the gridded position of each point in its exponent, for example the figure $\left\{\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{3}{2}, \frac{3}{2}\right)\right\}$ is in the equivalence class $1^{(0,0)}2^{(1,1)}$. A larger example is given in Figure 2.

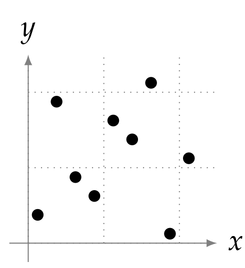

Figure 2: A drawing of a figure that is in the equivalence class represented by $2^{(0,0)}8^{(0,3)}4^{(1,1)}3^{(1,1)}7^{(2,3)}6^{(2,2)}9^{(3,4)}1^{(3,0)}5^{(4,2)}$.

The set of gridded permutations $\mathcal{G}$ is not a combinatorial class, as there are not finitely many gridded permutations of each length. In order to correct this we define $\mathcal{G}^{(n,m)}$ to be the set of gridded permutations with gridded positions in $\{0,\ldots,n-1\} \times \{0,\ldots,m-1\}$. This is a combinatorial class.

A gridded permutation *avoids* a set of gridded patterns $\mathcal{O}$ if it avoids every gridded permutation in $\mathcal{O}$. Let

$$\mathrm{Av}^{(n,m)}(\mathcal{O}) = \{\pi \in \mathcal{G}^{(n,m)} | \pi \text{ avoids } \mathcal{O}\}$$

be the *avoiders* of $\mathcal{O}$, and the $\mathrm{Av}_k^{(n,m)}(\mathcal{O})$ be the length k gridded permutations in this set. We say a gridded permutation *contains* a set of gridded patterns $\mathcal{R}$ if it does not avoid $\mathcal{R}$. That is it contains at least one gridded permutations in $\mathcal{R}$. The *containers* of $\mathcal{R}$ is the set

$$\mathrm{Co}^{(n,m)}(\mathcal{R}) = \{\pi \in \mathcal{G}^{(n,m)} | \pi \text{ contains } \mathcal{R}\}$$

and $\mathrm{Co}_k^{(n,m)}(\mathcal{R})$ is the set of length k gridded permutations in this set.

We define a tiling to be the triple $\mathcal{T} = ((n,m), \mathcal{O}, \mathcal{R} = \{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k\})$, where $\mathcal{O}$ is a set of gridded permutations called *obstructions* and $\mathcal{R}$ is a set of sets of gridded permutations that it must contain called *requirements*, i.e.

$$\mathcal{T} = \mathrm{Grid}\,(\mathcal{T}) = \mathrm{Av}^{(n,m)}(\mathcal{O}) \cap \mathrm{Co}^{(n,m)}(\mathcal{R}_1) \cap \mathrm{Co}^{(n,m)}(\mathcal{R}_2) \cap \cdots \cap \mathrm{Co}^{(n,m)}(\mathcal{R}_k).$$

For sake of brevity we will say $\mathcal{T} = \mathrm{Grid}\,(\mathcal{T})$, and refer to gridded permutations being on a tiling. Denote $\mathcal{T}_n$ to be the length n gridded permutations on $\mathcal{T}$.

Proposition 2. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two finite sets of gridded patterns. Then there exists a finite set $\mathcal{R}$ such that*

$$\mathrm{Co}^{(n,m)}(\mathcal{R}) = \mathrm{Co}^{(n,m)}(\mathcal{R}_1) \cap \mathrm{Co}^{(n,m)}(\mathcal{R}_2).$$

This result shows that containment of any number of sets of gridded permutations can be reduced to a single set, however it is often more convenient to think about requiring many smaller requirements.

We say a gridded permutation is *local* if the gridded positions of all the points are the same. We write π^c to represent the local gridded permutation with underlying pattern π with gridded positions being c.

Proposition 3. *Let $\mathcal{C} = \mathrm{Av}\,(\Pi)$ be a permutation class. Then the tiling*

$$\mathcal{T} = \Big((1,1), \{\sigma^{(0,0)} | \sigma \in \Pi\}, \emptyset\Big)$$

is in bijection with $\mathcal{C}$.

Proof. The mapping $\phi : \mathcal{C} \mapsto \mathcal{T}$ given by $\phi(\pi) = \pi^{(0,0)}$ is a bijection. □

The goal is to use the methods of the CombSpecSearcher to find combinatorial specifications for tilings. We will translate our geometric proof ideas for permutation classes into the language of gridded permutations and by using this proposition we will be able to enumerate permutation classes.

Translating proof techniques

We will illustrate how to translate standard methods to gridded permutations and tilings with a classic example from permutation patterns: how many permutations of length n avoid 231? The classic proof is as follows:

Proof. A permutation is either empty or not. A non-empty permutation $\pi \in \mathrm{Av}_n(231)$ must have some maximum element, so we can write it as $\pi = \alpha n \beta$. Given that π avoids 231, all the points in α must be below all the points in β. Moreover, if π avoids 231 then both α and β must avoid 231. This describes the structure of the permutations in $\mathrm{Av}_n(231)$. It can also be used to get the enumerative information from the class, that is $|\mathrm{Av}_n(231)|$ is given by n^{th} Catalan number. $\square$

We will now translate the proof methods from above to tilings and gridded permutations. From Proposition 3 we know that if we let $\mathcal{O} = \{231^{(0,0)}\}$ and $\mathcal{R} = \emptyset$ that the tiling $\mathcal{T} = ((1,1), \mathcal{O}, \mathcal{R})$ has a length preserving bijection to $\mathrm{Av}(231)$.

The first strategy we used was: either a permutation is empty or not. This idea can be captured by saying that a gridded permutation in $\mathcal{T}$ is either on the tiling $\mathcal{T}_1$ where $1^{(0,0)}$ is added to $\mathcal{O}$ or it is on the tiling $\mathcal{T}_2$ where the requirement $\{1^{(0,0)}\}$ is added to $\mathcal{R}$. Notice that $\mathcal{T}_1$ consists only of the empty gridded permutation. This can be generalized to allow adding longer requirements, and collectively we call this strategy *requirement insertion*. Cleary, it is a combinatorial rule with the admissable constructor disjoint union.

The second strategy was to write a non-empty permutation π as $\alpha n \beta$. We represent this by creating a tiling, $\mathcal{T}_3$ with dimensions $(2,3)$ which we can have a length preserving bijection to. To capture the correct strategy we add the requirement $\{1^{(1,1)}\}$ and to the obstructins we add $1^{(0,1)}$, $1^{(1,0)}$ and $1^{(2,1)}$ which force these cells to be empty. We also add the obstructions $12^{(1,1)}$ and $21^{(1,1)}$, which ensures cell $(1,1)$ contains exactly a point, so representing n. We then add the obstructions that can occur across the two remaining opening cells that represent α and β, to ensure that the tiling avoids 231. This strategy we call *point placement* and is a combinatorial rule, but more, it shows equivalence between $\mathcal{T}_2$ and $\mathcal{T}_3$.

In $\mathcal{T}_3$ we can ignore some redundant obstructions, since all of the crossing obstructions contain the smaller gridded permutation $2^{(0,0)}1^{(2,0)}$. This length 2 obstruction tells you that all of the points in cell $(0,0)$ in the gridded permutations on $\mathcal{T}_3$ appear below all of the points in cell $(2,0)$. We can represent this by creating another tiling, $\mathcal{T}_4$, that is 3×3 and essentially puts these two cells on their own row. We call this inferral strategy *row separation*. This is an equivalence between $\mathcal{T}_3$ and $\mathcal{T}_4$.

The final strategy we use is to observe that since $\mathcal{T}_4$ has no crossing obstructions and all the non-empty cells are on separate rows and columns, the gridded permutations can be made by picking the points in each cell independently. Therefore, we can think of it as the Cartesian product of the 3 non-empty cells as 1×1 tilings. We call this strategy *factors*, and clearly, this is a combinatorial rule with the admissable constructor Cartesian product.

We have shown a pictorial representation of this combinatorial specification using these strategies in Figure 1. The obstructions are shown in red, and the requirements in green. The cell that is a point is represented by a black point.

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The Local Structure of Permutations with Given Inversion Density

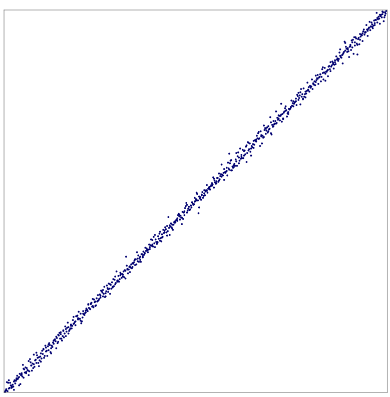
David Bevan University of Strathclyde

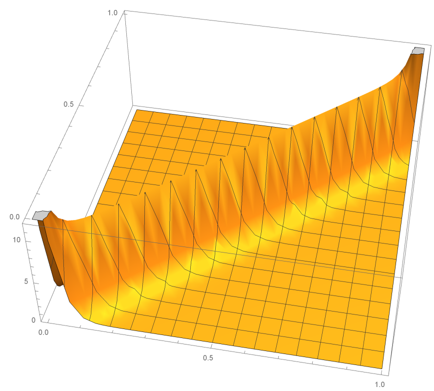
This talk is based on joint work with Thomas Selig

At the left is a typical permutation on 750 points with 2809 inversions. The probability that two points chosen randomly from this permutation form an inversion is very close to 1%.

Question. What is the probability that two *adjacent* points form an inversion (a *descent*) in a random n-point permutation with $\lfloor \binom{n}{2}/100 \rfloor$ inversions, as $n \to \infty$?

A *permuton*, like that at the right, precisely describes the distribution of points in a random permutation with fixed inversion density (see [1]). Can this help answer the question?





We will investigate the extent to which permutons can (and cannot) be used to answer this question, and look at what more can be said about the local structure of a large random permutation with specified inversion density. We may also consider what happens if the number of inversions grows more slowly than any fixed proportion of the maximum possible.

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Patterns in Standard Young Tableaux

Sara Billey University of Washington

This talk is based on joint work with Matjaž Konvalinka and Joshua Swanson

Standard Young tableaux are fundamental in combinatorics, representation theory, and geometry. The major index statistic originally defined for permutations has been extended to tableaux and can be interpreted in each of these settings. We consider the probability distribution of the major index on standard tableaux of fixed partition shape chosen uniformly along with the corresponding generating function.

We give an explicit hook length formula for all of the cumulants of these distributions using recent work of Chen–Wang–Wang. The cumulant formula allows us to classify all possible limit laws for any sequence of shapes in terms of a simple auxiliary statistic, aft, generalizing earlier results of Canfield–Janson–Zeilberger, Chen–Wang–Wang, and others. We show that any such sequence of distributions with aft approaching infinity is asymptotically normal.

This leads to a series of questions concerning locations of zero coefficients, unimodality, and asymptotic estimates for the major index generating functions over all standard tableaux of a fixed shape. We settle the first of these by identifying mutation rules in terms of tableaux patterns leading to both a strong and weak poset structure on tableaux ranked by the major index. The classification of zeros can be interpreted as determining which irreducible representations of the symmetric group exist in each homogeneous components of the corresponding coinvariant algebra. We give conjectured answers concerning unimodality and asymptotic estimates.

Rook and Wilf Equivalence of Integer Partitions

Jonathan Bloom Lafayette College

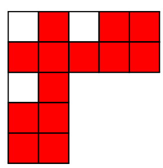
This talk is based on joint work with Nathan McNew and Dan Saracino

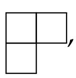
In this talk we introduce a notion of pattern avoiding integer partitions and discuss how this idea is intimately connected to the well studied area of Rook Theory. In particular, our main result is that Wilf-equivalence (under integer partition avoidance) is the same as Foata and Schützenberger's notion of rook equivalence. Lastly, and if time permits, we will look at a family of integer partition patterns that yield rational generating functions and discuss some corollaries of this result. To formally state and describe these results some definitions are needed.

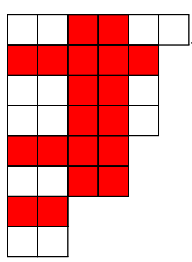
Let $\mathcal{P}$ be the set of all integer partitions and refine this for any $n \geq 0$ by letting $\mathcal{P}_n$ be the set of all integer partitions μ whose *weight* (i.e., the sum of the parts) is n. We denote the weight of a partition μ by $|\mu|$. For any partition μ we define its *height* to be the number of positive parts h_μ. Likewise, we define its *width*, which we denote by w_μ, to be the size of its first part μ_1. Using these notions we also refine our set $\mathcal{P}$ by defining, for any $h, k \geq 0$, the set

$$\mathcal{P}(h,k) = \{\mu \in \mathcal{P} \,:\, h_\mu \leq h, w_\mu = k\}.$$

Our notion of pattern avoidance becomes natural when we view integer partitions as Ferrers boards. In particular, we say a partition α *contains* a partition μ provided that it is possible to delete rows and columns from α so that one obtains μ. For example, the partition

$(5,5,2,2,2) =$ 

contains $(2,1) =$ , since deleting the colored rows and columns yields $(2,1)$. For another example, if $\mu = (4,3,3,2,2)$ then by deleting the indicated rows and columns we see that μ is contained in the partition

.

We define $\mathcal{P}(\mu)$ to be the set of all partitions that contain μ and we define $\mathcal{P}_n(\mu) = \mathcal{P}(\mu) \cap \mathcal{P}_n$. We say that $\mu, \tau \in \mathcal{P}$ are *Wilf equivalent* provided that

$$|\mathcal{P}_n(\mu)| = |\mathcal{P}_n(\tau)|$$

for all $n \geq 0$. Similarly, we say that μ and τ are *width-Wilf equivalent* provided that there are the same number of partitions of each weight and width that contain μ as there are that contain τ.

Further we define for any $k \geq 0$

$$\mathcal{P}(\mu, k) = \{\alpha \in \mathcal{P}(\mu), : w_\alpha = k + w_\mu\}$$

and $\mathcal{P}_n(\mu, k) = \mathcal{P}_n \cap \mathcal{P}(\mu, k)$ and set

$$F_{\mu,k}(q) = \sum_{n \geq 0} |\mathcal{P}_n(\mu, k)| q^n.$$

Lastly, we also need to recall some basic rook theory. We say two integer partitions (when viewed as Ferrers boards) are *rook equivalent* provided that they admit the same number of k-rook placements for all $k \geq 0$. In [3], Foata and Schützenberger give a beautiful "multiset" criterion for rook equivalence. They prove that μ and ν are rook equivalent if and only if the multisets

$$\{i + \mu_i, : i \leq h\} = \{i + \nu_i, : i \leq h\}$$

are equal. (In this case we add empty rows to one of them if necessary, so that they have the same number of rows.)

Our main contribution is a new (and unexpected) characterization of the old idea of rook equivalence. In particular, we establish the following theorem.

Theorem 1 (Bloom, Saracino). *Partitions μ and τ are rook equivalent if and only if they are Wilf equivalent.*

The proof of the forward direction can be found in the first paper on this topic, see [2]. Although this proof is long and technical the main idea is to show that the generating function $F_{\mu,k}(q)$ can be described in terms of the multiset $\{i + \mu_i, : i \leq h\}$. On the other hand, the converse is the main result of the second paper on this topics, see [1]. To prove this direction, we show that for two distinct decreasing partitions of n, μ and ν we have

$$|\mathcal{P}_{n+m-1}(\mu)| < |\mathcal{P}_{n+m-1}(\nu)|$$

where $m = r + \nu_r$ and r is the largest integer such that $\mu_r \neq \nu_r$. We note that showing this for decreasing partitions is sufficient as the set of such partitions is a transversal on the set of rook equivalence classes.

Additionally, in [2], we also characterize the notion of width-Wilf equivalence.

Theorem 2 (Bloom, Saracino). *For any nonempty partitions μ and τ that have the same weight and width, the following are equivalent:*

(i) *μ and τ are width-Wilf equivalent*

(ii) $F_{\mu,1} = F_{\tau,1}$

(iii) *μ and τ are rook equivalent.*

Most recently, Bloom and McNew have looked at characterizing the generating functions

$$G_\mu(z) = \sum_{n\geq 0} |\mathcal{P}_n \setminus \mathcal{P}_n(\mu)| z^n,$$

i.e., the generating function for pattern *avoiding* partitions. (Recall that $\mathcal{P}_n(\mu)$ is the set of partitions with weight n that *contain* μ.)

Theorem 3 (Bloom, McNew)**.** *Let μ be a partitions whose parts differ in length by at least 2. Then $G_\mu(z)$ is rational.*

We prove this theorem by developing an algorithm that recursively computes these generating functions for such μ. As a corollary of this theorem we are able to prove two curious facts.

Corollary 4 (Bloom, McNew)**.** *Fix some $M \geq 1$ and let $\mathcal{Q}_n(M)$ be the set of all $\mu \in \mathcal{P}_n$ so that $|\mu_i - \mu_j| < M$ for all $i, j \leq h_\mu$. Then $\sum_{n\geq 0} |\mathcal{Q}_n(M)| z^n$ is rational.*

Via numerical evidence and the OEIS sequence A136185 we speculated that the generating function for $\mu = (5,2)$

$$1 + 2z + 5z^2 + 7z^3 + 11z^4 + 14z^5 + 20z^6 + \cdots$$

was equal to the generating function for the number of metacyclic groups of prime order as found in [4]. Using our algorithm to construct this rational generating function we found it to be

$$G_{(5,2)}(z) = \frac{-z(z^7 - 2z^5 + z^3 + z^2 - z - 1)}{(z-1)^4(z+1)^2(z^2+z+1)},$$

which is precisely the same one found for metacyclic groups of prime order. This gives us the following result.

Corollary 5 (Bloom, McNew)**.** *The partitions that avoid $(5,2)$ are in bijection with metacyclic groups of prime order.*

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Local Convergence for Random Permutations and the Case of Uniform ρ-Avoiding Permutations with $|\rho| = 3$

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For large combinatorial structures, two main notions of convergence can be usually defined: scaling limits and local limits. Informally scaling limits consist in studying the objects from a global point of view (after a rescaling of the distances between points of the objects), while for local limits we study the objects in a neighborhood around a marked point (without rescaling distances). In particular for graphs, both notions are well-studied and well-understood. For permutations only a notion of scaling limits, called permutons, has been recently introduced (see [5]). The convergence in the sense of permutons has also been characterized by frequencies of pattern occurrences (see [2]).

Our main results can be divided into two different parts:

1. We set up a new notion of local convergence for permutations and we prove a characterization in terms of proportions of *consecutive* pattern occurrences. We are also able to characterize random limiting objects introducing a "shift-invariant" property.

2. We show examples of local convergence in the framework of random pattern-avoiding permutations: we describe the asymptotics in n of the number of consecutive occurrences of any fixed pattern π in a uniform ρ-avoiding permutation of size n, for $|\rho| = 3$. For this last result we use bijections between ρ-avoiding permutations and ordered rooted trees and singularity analysis.

Local convergence for permutations

This section is inspired by local convergence for trees (see *e.g.* [1]) and Benjamini-Schramm convergence for random graphs (see [3]). In the context of local convergence we need to look at permutations with a marked entry, called root. We denote with $\mathcal{S}^n$ the set of permutations of size n and with $\mathcal{S}$ the set of permutations of finite size.

Definition 1. A *finite rooted permutation* is a pair (σ, i), where $\sigma \in \mathcal{S}^n$ and $i \in [n]$.

To a rooted permutation (σ, i), we associate (as shown in the left hand-side of Fig. 1) the pair $(A_{\sigma,i}, \preceq_{\sigma,i})$, where $A_{\sigma,i} := [-i+1, |\sigma| - i]$ is a finite interval of integers containing 0 and $\preceq_{\sigma,i}$ is a total order on $A_{\sigma,i}$, defined for all $\ell, j \in A_{\sigma,i}$ by $\ell \preceq_{\sigma,i} j$ if $\sigma_{\ell+i} \leq \sigma_{j+i}$.

Given the diagram of a rooted permutation (σ, i), the corresponding total order $(A_{\sigma,i}, \preceq_{\sigma,i}$ $)$ is simply obtained by shifting the origin of the x-axis to the column containing the root of the permutation (the new indices are reported under the columns of the diagram) and then reading the diagram of the permutation from bottom to top, gradually recording, for each element, its position according to the new x-axis.

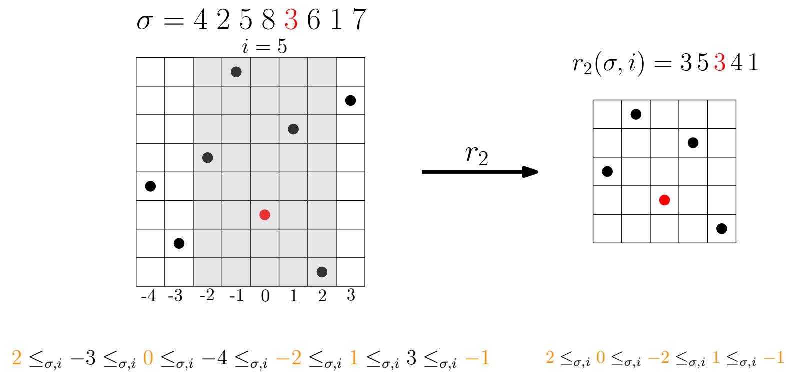

Figure 1: Restriction function for rooted permutations.

Since the map from the space of finite rooted permutations to the space of total orders on finite integer intervals containing zero is a bijection, we identify every rooted permutation (σ, i) with the total order $(A_{\sigma,i}, \preceq_{\sigma,i})$.

Thanks to this identification, the following definition of infinite rooted permutation is natural.

Definition 2. We call *infinite rooted permutation* a pair $(A, \preceq)$ where A is an infinite interval of integers containing 0 and $\preceq$ is a total order on A.

The next step is to define a notion of local convergence in the space of rooted (finite or infinite) permutations denoted by $\tilde{\mathcal{S}}_\bullet$. In order to do that we introduce a notion of neighborhoods around the root which can be thought as a "vertical strip" around the root of the permutation (see Fig. 1). Formally, for all $h \in \mathbb{N}$, we define the *restriction function around the root* as

$$\begin{aligned} r_h \colon \quad \tilde{\mathcal{S}}_\bullet &\longrightarrow \quad \tilde{\mathcal{S}}_\bullet \\ (A, \preceq) &\mapsto (A \cap [-h, h], \preceq) \,. \end{aligned}$$

The restriction r_h of the diagram around the root coincide with the diagram of the rooted permutation induced by the pattern $\text{pat}_{[a,b]}(\sigma)$ where $a = \max\{i - h, 1\}$ and $b = \min\{i + h, |\sigma|\}$.

Example 3. We refer to Fig. 1. On the top of the picture we see the restriction $r_2(\sigma, i)$ from the diagram point of view and on the bottom from the total order point of view.

Definition 4. We say that a sequence $(A_n, \preceq_n)_{n\in\mathbb{N}}$ of rooted permutations in $\tilde{\mathcal{S}}_\bullet$ is *locally convergent* to an element $(A, \preceq) \in \tilde{\mathcal{S}}_\bullet$, if for all $h > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$r_h(A_n, \preceq_n) = r_h(A, \preceq).$$

This topology is metrizable by a local distance d and we prove that the space $(\tilde{\mathcal{S}}_\bullet, d)$ is a compact Polish space.

We have defined a notion of local convergence for rooted permutations and we want to extend this notion to study sequences of (unrooted) permutations. We can see a fixed permutation σ as a rooted object only after a root i has been chosen. A natural way to choose a root is to make the choice at random, and uniformly among the indices of σ. In this way, a fixed permutation σ naturally identifies a random variable $(\sigma, \boldsymbol{i})$ that takes values in the set of finite rooted permutations (we denote random quantities using **bold** characters).

Definition 5. Given a sequence $(\sigma_n)_{n\in\mathbb{N}}$ of elements in $\mathcal{S}$, we say that $(\sigma_n)_{n\in\mathbb{N}}$ *Benjamini-Schramm converges* to a random (possibly infinite) rooted permutation $(\boldsymbol{A}, \preccurlyeq)$, if the sequence $(\sigma_n, \boldsymbol{i}_n)_{n\in\mathbb{N}}$, where $\boldsymbol{i}_n$ is a uniform index in $[1, |\sigma_n|]$, converges in distribution to $(\boldsymbol{A}, \preccurlyeq)$ with respect to the local distance d.

We prove the following characterization in terms of proportions of consecutive pattern occurrences. For any pattern π of size k, and any permutation σ of size n, we denote by

$$\widetilde{c\text{-}occ}(\pi, \sigma) = \frac{\text{number of consecutive occurrences of } \pi \text{ in } \sigma}{n},$$

the proportion of consecutive occurrences of π in σ.

Theorem 6. *For any $n \in \mathbb{N}$, let σ^n be a permutation of size n. Then the Benjamini-Schramm convergence for the sequence $(\sigma^n)_{n\in\mathbb{N}}$ is equivalent to the existence of an infinite vector of non-negative real numbers $(\Delta_\pi)_{\pi\in\mathcal{S}}$ such that, for all patterns $\pi \in \mathcal{S}$,*

$$\widetilde{c\text{-}occ}(\pi, \sigma^n) \to \Delta_\pi.$$

We then extend, in two non-equivalent ways, the above notion of Benjamini-Schramm convergence to sequences of *random* permutations $(\boldsymbol{\sigma}^n)_{n\in\mathbb{N}}$ introducing the *annealed* and the *quenched* version of the Benjamini-Schramm convergence. We obtain the following two characterizations:

Theorem 7. *For any $n \in \mathbb{N}$, let $\boldsymbol{\sigma}^n$ be a random permutation of size n. Then*

(a) The annealed version of the Benjamini-Schramm convergence of $(\boldsymbol{\sigma}^n)_{n\in\mathbb{N}}$ is equivalent to the existence of an infinite vector of non-negative real numbers $(\Delta_\pi)_{\pi\in\mathcal{S}}$ such that for all patterns $\pi \in \mathcal{S}$,

$$\mathbb{E}[\widetilde{c\text{-}occ}(\pi, \boldsymbol{\sigma}^n)] \to \Delta_\pi.$$

(b) The quenched version of the Benjamini-Schramm convergence of $(\boldsymbol{\sigma}^n)_{n\in\mathbb{N}}$ is equivalent to the existence of an infinite vector of non-negative real random variables $(\boldsymbol{\Lambda}_\pi)_{\pi\in\mathcal{S}}$ such that

$$\left(\widetilde{c\text{-}occ}(\pi, \boldsymbol{\sigma}^n)\right)_{\pi\in\mathcal{S}} \stackrel{(d)}{\to} (\boldsymbol{\Lambda}_\pi)_{\pi\in\mathcal{S}},$$

w.r.t. the product topology (where $\stackrel{(d)}{\to}$ indicates the convergence in distribution).

Obviously, the quenched version implies the annealed version.

Finally we are also able to characterize random limiting objects for the annealed version of the Benjamini-Schramm convergence introducing a "shift-invariant" property (corresponding to the well-known notion of unimodularity for random graphs).

Local convergence for uniform ρ-avoiding permutations with $|\rho| = 3$

We show the relevance of this new notion of convergence proving the local convergence for random ρ-avoiding permutations and characterizing their limits.

Theorem 8. *Let $\rho \in \mathcal{S}^3$ and for any $n \in \mathbb{N}$, let $\boldsymbol{\sigma}^n$ be a uniform random ρ-avoiding permutation. Then we have the following convergence in probability,*

$$\widetilde{c\text{-}occ}(\pi, \boldsymbol{\sigma}^n) \xrightarrow{P} P_\rho(\pi), \quad \textit{for all} \quad \pi \in \mathrm{Av}(\rho), \tag{1}$$

where for all $m \in \mathbb{N}$, $\big(P_\rho(\pi)\big)_{\pi \in \mathrm{Av}^m(\rho)}$ is a probability distribution on $\mathrm{Av}^m(\rho)$ given below.

An interesting aspect of the theorem is the condensation phenomenon, indeed the limits of the random sequences $\big(\widetilde{c\text{-}occ}(\pi, \boldsymbol{\sigma}^n)\big)_{n \in \mathbb{N}}$ are deterministic, for all pattern $\pi \in \mathcal{S}$. This also implies that the convergence in probability in Equation (1) is equivalent to the convergence in distribution of the vector $\big(\widetilde{c\text{-}occ}(\pi, \boldsymbol{\sigma}^n)\big)_{\pi \in \mathcal{S}}$. Therefore our theorem trivially implies the characterization (b) in Theorem 7 and so prove that the sequence $(\boldsymbol{\sigma}^n)_{n \in \mathbb{N}}$ converge for the quenched version of the Benjamini-Schramm convergence. Moreover, we are able to provide a construction of the limiting random order on $\mathbb{Z}$.

We now present the main ideas of the proof of Theorem 8. First of all we note that it is enough to analyze the two cases $\rho = 321$ and $\rho = 231$. The other four cases are obtained using symmetries of the square.

We explicitly exhibit the probability distributions

$$\big(P_{231}(\pi)\big)_{\pi \in \mathrm{Av}^m(231)}$$

and

$$\big(P_{321}(\pi)\big)_{\pi \in \mathrm{Av}^m(321)}$$

that appear in the statement of the theorem. For all $m \geq 0$,

$$P_{231}(\pi) := \frac{2^{\mathrm{LRMax}(\pi) + \mathrm{RLMax}(\pi)}}{2^{2|\pi|}}, \quad \text{for all} \quad \pi \in \mathrm{Av}(231),$$

where $\mathrm{LRMax}(\pi)$ (resp. $\mathrm{RLMax}(\pi)$) denotes the number of left-to-right maxima (resp. right-to-left maxima) in π. Moreover, for all $m \geq 0$ and for all $\pi \in \mathrm{Av}^m(231)$,

$$P_{321}(\pi) := \begin{cases} \frac{|\pi|+1}{2^{|\pi|}} & \text{if } \pi = 12\ldots|\pi|, \\ \frac{1}{2^{|\pi|}} & \text{if } c\text{-}occ(21, \pi^{-1}) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We underline that the two limiting distributions present an important difference: the first one has full support on $\mathrm{Av}^m(231)$, whereas the second gives positive measure only to 321-avoiding permutations whose inverse have at most one descent.

We give some hints into the proof in the case $\rho = 231$. The proof is divided into two main steps:

<u>FIRST STEP:</u> The goal is to prove that $\mathbb{E}[\widetilde{c\text{-}occ}(\pi, \boldsymbol{\sigma}^n)] \to P_{321}(\pi)$, for all $\pi \in \mathrm{Av}(231)$.

We use a technique introduced by Janson [6]. Using a well-known bijection between 231- avoiding permutations and binary trees (see [4]), instead of considering a sequence of uniform 231-avoiding permutations of size n, we can consider a sequence of uniform binary trees $\boldsymbol{T}_n$ with n vertices. The behavior of $\boldsymbol{T}_n$ can be understood considering a binary Galton-Watson tree $\boldsymbol{T}_\delta$ with offspring distribution $\eta(\delta)$, $\delta \in (0,1)$. Indeed, for some specific distribution $\eta(\delta)$, the law of a binary Galton-Watson tree $\boldsymbol{T}_\delta$ conditioned on having n vertices is equal to the law of $\boldsymbol{T}_n$. Using this result it is possible to relate $\boldsymbol{T}_\delta$ and the sequence $(\boldsymbol{T}_n)_{n\in\mathbb{N}}$ by the formula

$$\mathbb{E}\big[F(\boldsymbol{T}_\delta)\big] = \frac{1+\delta}{1-\delta}\sum_{n=1}^{+\infty}\mathbb{E}\big[F(\boldsymbol{T}_n)\big]\cdot C_n\cdot\Big(\frac{1-\delta^2}{4}\Big)^n, \quad \text{for all bounded functions } F. \quad (2)$$

With a recursive proof we show that

$$\mathbb{E}\big[c\text{-}occ(\pi, \boldsymbol{T}_\delta)\big] = \delta^{-1}\cdot P_{231}(\pi) + P(\delta),$$

where $P(\delta)$ is a polynomial in δ. Then applying singularity analysis to the function in Equation (2) (which is Δ-analytic in $z(\delta) = \frac{1-\delta^2}{4}$) we conclude that

$$\mathbb{E}\big[\widetilde{c\text{-}occ}(\pi, \boldsymbol{T}_\delta)\big] \to P_{231}(\pi), \quad \text{for all} \quad \pi \in \mathrm{Av}(231).$$

We finally conclude the proof going back to 231-avoiding permutations, working backwards the above mentioned bijection with trees"

<u>SECOND STEP</u>: The goal is to prove $\widetilde{c\text{-}occ}(\pi, \boldsymbol{\sigma}^n) \xrightarrow{P} P_{231}(\pi)$, for all $\pi \in \mathrm{Av}(231)$.

We study the second moment $\mathbb{E}[\widetilde{c\text{-}occ}(\pi, \boldsymbol{\sigma}^n)^2]$ using similar techniques as before and we obtain

$$\mathbb{E}\big[c\text{-}occ(\pi, \boldsymbol{T}_\delta)^2\big] \to P_{231}(\pi)^2, \quad \text{for all} \quad \pi \in \mathrm{Av}(231).$$

We conclude noting that

$$\mathrm{Var}\big(c\text{-}occ(\pi, \boldsymbol{T}_\delta)^2\big) \to 0, \quad \text{for all} \quad \pi \in \mathrm{Av}(231),$$

and applying the second moment method.

The proof in the case $\rho = 321$ is not presented in this abstract and uses different techniques.

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The Undecidability of the Joint Embedding Property for Finitely-Constrained Hereditary Graph Classes

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We prove that there is no algorithm that, given a finite set of forbidden induced subgraphs, decides whether the corresponding hereditary graph class has the joint embedding property. This may be a step toward proving there is no algorithm that, given a finite set of forbidden permutation patterns, decides whether the corresponding pattern avoidance class is atomic.

A pattern avoidance class is called *atomic* if it cannot be written as a union of two proper subclasses. Pattern avoidance classes can sometimes be understood by understanding their atomic subclasses, as in the following result.

Proposition 1 ([3]). *Let $\mathcal{C}$ be a pattern avoidance class without an infinite antichain in the containment order. Then $\mathcal{C}$ can be written as a finite union of atomic subclasses, and the upper growth rate of $\mathcal{C}$ is equal to the greatest upper growth rate of its atomic subclasses.*

In his work on homogeneous permutations [1], Cameron introduced the structural view of permutations as structures in a language of two linear orders. A permutation π is contained in σ exactly when, from this structural viewpoint, π embeds into σ. Atomicity is then seen to be equivalent to the following standard model-theoretic property.

Definition 2. A class of structures $\mathcal{C}$ has the *joint embedding property (JEP)* if for all $A, B \in \mathcal{C}$, there exists a $C \in \mathcal{C}$ embedding both.

Proposition 3 ([3]). *Let $\mathcal{C}$ be a pattern avoidance class. Then $\mathcal{C}$ is atomic iff $\mathcal{C}$ has the JEP.*

In [2], Ruškuc posed the following question.

Question 4. *Is there an algorithm that, given finite set of forbidden permutations, decides whether the corresponding pattern avoidance class is atomic (equivalently, has the JEP)?*

This problem is known to be decidable in certain restricted classes of permutations, such as grid classes [4]. However, we believe there is a strong possibility this decision problem is undecidable in general. As a first approximation, we examine the corresponding problem in the category of graphs. Here we obtain the following, via a reduction to the tiling problem.

Theorem 5. *There is no algorithm that, given a finite set of forbidden induced subgraphs, decides whether the corresponding hereditary graph class has the JEP.*

In the tiling problem, one is given a finite set of tile types, and is asked whether a grid can be tiled, subject to constraints that certain tile-types cannot be horizontally or vertically adjacent to certain others.

A very rough sketch of the proof our theorem is as follows. The first two steps ensure that the tiling problem is equivalent to whether we can joint-embed two particular graphs, and the third step ensures that joint-embedding for the class is equivalent to joint-embedding for those two graphs.

1. Construct two graphs A^*, representing a grid, and B^* representing a suitable collection of tiles.
2. Choose a finite set of constraints to ensure that successfully joint-embedding A^* and B^* requires producing a valid tiling of the grid points in A^* with the tiles from B^*
3. Show that if the tiling problem admits a solution, then the chosen class admits a joint-embedding procedure.

We are hopeful that suitable coding techniques will allow this result to be brought over to the category of permutations, although this currently seems more promising for higher-dimensional permutations, i.e. structures in a language of 3 or more linear orders.

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Permutation Classes and Infinite Antichains

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Given some suitably defined permutation class (or family of permutation classes), notable questions one might like answers to include: How many permutations of length n are there? What is the growth rate of the class? Is it finitely based? What do the permutations 'look' like? Although rather more vague than the others, the last question in that list – asking for a structural characterization – can underpin the other three. Here are some recent examples: (1) the drive by multiple authors to enumerate the '2×4' classes, (2) bounding the growth rate of Av(1324), and (3) in the study of grid classes, 'geometric' ones are (among other nice properties) finitely based.

Here's another question: Does the class contain infinite antichains? Unless you're similarly-minded to me, it's quite likely that this question wasn't high on your list. Nevertheless, in the last 10 years several authors (not including me) have established that the existence of one particular infinite antichain, the 'increasing oscillating antichain', has a profound impact on the theory of permutation classes: It tells us that $\kappa \approx 2.206$ is the smallest growth rate where there are uncountably many permutation classes, and below κ it helps to determine what the possible growth rates are; it also guarantees that any real number about $\lambda \approx 2.357$ is the growth rate of some permutation class. (In fact, the underpinning 'oscillation' structure has been known to mathematical biologists at least since 1991 under the term 'Gollan permutation', as the unique (up to symmetry) hardest permutation to sort by reversals.)

With the aid of a variant of permutation containment called 'labelled containment', in this talk I will survey instances where the existence of — or (more often) lack of — infinite antichains in a permutation class interacts with these other questions. I will report on recent joint work with David Bevan and Nik Ruškuc in the study of monotone grid classes, where a result on classifying the infinite antichains also tells us that every 'unicyclic' monotone grid class must be finitely based.

Some Relations on Prefix Reversal Generators of the Symmetric and Hyperoctahedral Group

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This talk is based on joint work with Saúl A. Blanco

In this extended abstract, we describe some relations satisfied by prefix-reversal generators of the symmetric groups. We also describe all relations satisfied by two generators in the group of signed permutations and draw connections with cycles in the pancake graph (Cayley graph with vertex set permutations or signed permutations using the prefix-reversal as generators).

Introduction

The first appearance of the pancake problem in print was in the Problems and Solutions section of the December 1975 *Monthly* [8].

> The chef in our place is sloppy, and when he prepares a stack of pancakes they come out all different sizes. Therefore, when I deliver them to a customer, on the way to the table I rearrange them (so that the smallest winds up on top, and so on, down to the largest on the bottom) by grabbing several from the top and flipping them over, repeating this (varying the number I flip) as many times as necessary. If there are n pancakes, what is the maximum number of flips (as a function $f(n)$ of n) that I will ever have to use to rearrange them?

The problem of determining the maximum number of flips that are ever needed to sort a stack of n pancakes is known as the *pancake problem*, and the $f(n)$ is known as the *pancake number*.

This initial posing of the problem was made by Goodman under the pseudonym Harry Dweighter (a pun for “harried waiter"). In [9], as a commentary to the problem formulation in [8], Garey, Johnson, and Lin gave the first upper and lower bound to the the pancake number:

$$n + 1 \leq f(n) \leq 2n - 6 \text{ for } n \geq 7.$$

Subsequent results have been successful in tightening these bounds. The first significant tightening of the bounds was described in the work of Gates and Papadimitriou [11], which incidentally is the only academic paper Gates ever wrote. The best upper and lower bound known today for the general case appeared in [5] and [12], respectively. Combined, one has that

$$15\left\lfloor \frac{n}{14} \right\rfloor \leq f(n) \leq \frac{18n}{11} + O(1).$$

Computing the pancake number for a given n is a complicated task. To our knowledge, the exact value of $f(n)$ is only known for $1 \leq n \leq 19$ (see [2, 6, 7, 12, 16]). In fact, determining the minimum number needed to sort a stack of pancakes is an NP-hard problem [4], though 2-approximation algorithms exists [10].

The pancake problem has connections to parallel computing, in particular in the design of symmetric interconnection networks (networks used to route data between the processors in a multiprocessor computing system) where the so-called *pancake graph*, the Cayley graph of the symmetric group under prefix reversals, gives a model for processor interconnections (see [1, 18]). One can also define a *burnt pancake graph* on signed permutations.

Terminology and Notation

Let S_n be the group of permutations of the set $[n] := \{1, 2, \dots, n\}$ and denote by e the identity permutation. The group S_n is generated by the set $S := \{s_1, \dots, s_{n-1}\}$ of adjacent transpositions; that is, $s_i = (i, i+1)$ in cycle notation. The set S is subject to the relations $s_i^2 = e$ for all $1 \leq i \leq n-1$, $(s_i s_{i+1})^3 = e$ for all $1 \leq i \leq n-2$, and $s_i s_j = s_j s_i$ for all $i, j \in [n-1]$ with $|i - j| \geq 2$.

The pancake problem has a straight-forward interpretation in terms of permutations. A stack of n pancakes of different sizes can be thought of as an element of S_n and flipping a stack of pancakes with a spatula can be thought of as using a *prefix reversal permutation*; that is, a permutation whose only action when composed with $w \in S_n$ is to reverse the first so many characters of w, in one-line notation. In other words, using one-line notation, a prefix reversal permutation of S_n has the form

$$\begin{aligned} &(i+1)\ \ i\ (i-1)\ \ \cdots 2\ \ 1\ \ (i+2)\ \ (i+3)\ \ \cdots n \\ &= (1, i+1)(2, i) \cdots \left(\left\lfloor \frac{i+2}{2} \right\rfloor, \left\lceil \frac{i+2}{2} \right\rceil \right), \text{ as a product of transpositions,} \end{aligned}$$

for some $i \in [n-1]$.

We denote the above permutation by f_i, with $1 \leq i \leq n-1$ and define $P = \{f_1, \dots, f_{n-1}\}$. For example, in S_4 one has $f_1 = 2134, f_2 = 3214$, and $f_3 = 4321$. Notice that effect of applying f_i to a permutation is similar to that of using a spatula to flip a stack of pancakes since one is reversing the order of the first $i+1$ entries and leaving the rest untouched.

One can easily see that $s_i = f_i f_1 f_i$ and that $f_i = s_1 \dots s_{i-1} \dots s_2 s_1 s_i \dots s_2 s_1$. Hence, S_n is also generated by P. We refer to the elements of P as *pancake generators* of S_n.

Let B_n be the *hyperoctahedral group*, most commonly referred to as the group of *signed permutations* of the set $[\pm n] = \{\underline{n}, \underline{n-1}, \dots, \underline{1}, 1, 2, \dots, n\}$, where $\underline{i} = -i$. That is, permutations w of $[\pm n]$ satisfying $w(\underline{i}) = \underline{w(i)}$ for all $i \in [\pm n]$. We shall use *window notation* to denote $w \in B_n$; that is, we denote w by $[w(1)\ w(2)\ \dots\ w(n)]$. The group B_n is generated by the set $\{s_0^B, s_1^B, \dots, s_{n-1}^B\}$, where $s_0^B = [\underline{1}\ 2\ \cdots\ n]$ and for $1 \leq i \leq n-1$,

$s_i^B = [1\ 2\ \cdots (i-1)\ (i+1)\ i\ (i+2) \cdots\ n]$ (see [3, Chapter 8]).

The burnt pancake generators affect the orientation of the entries: they are negative if they have been reversed an odd number of times and positive otherwise. We define f_i^B, $1 \le i \le n-1$ to be the signed permutation

$$f_i^B = [\underline{i+1}\ \underline{i}\ \underline{i-1}\ \cdots \underline{2}\ \underline{1}\ (i+2)\ (i+3)\ \cdots n]$$
$$= (1, \underline{i+1}, \underline{1}, i+1)(2, \underline{i}, \underline{2}, i) \cdots \left(\left\lfloor \frac{i+2}{2} \right\rfloor, \underline{\left\lceil \frac{i+2}{2} \right\rceil}, \underline{\left\lfloor \frac{i+2}{2} \right\rfloor}, \left\lceil \frac{i+2}{2} \right\rceil \right)$$

in disjoint cycle form as elements of the symmetry group of $[\pm n]$,

and $f_0^B = s_0^B$. Thus, for example, in B_4 we have $f_0^B = [\underline{1}\ 2\ 3\ 4], f_1^B = [\underline{2}\ \underline{1}\ 3\ 4], f_2^B = [\underline{3}\ \underline{2}\ \underline{1}\ 4]$, and $f_3^B = [\underline{4}\ \underline{3}\ \underline{2}\ \underline{1}]$. We shall define $P^B = \{f_0^B, f_1^B, \ldots, f_{n-1}^B\}$ as the set of *burnt pancake generators*, or *burnt pancake flips*.

One can see that $s_i^B = f_i^B f_0^B f_1^B f_0^B f_i^B$ for $1 \le i \le n-1$ and $s_0^B = f_0^B$, thus B_n is also generated by P^B. Furthermore, we note that $f_i^B = s_0^B s_1^B \ldots s_0^B s_{i-1}^B \ldots s_2^B s_1^B s_0^B s_i^B \ldots s_2^B s_1^B s_0^B$.

Results

S_n results

In this section we take a look at the pancake generators for S_n. In this case, the *pancake matrix*, $M_{n-1} = (m_{i,j})_{(n-1)\times(n-1)}$, where $m_{i,j}$ is the order of $f_i f_j$. The theorem below provides a description for M_{n-1}. It turns out M_{n-1} is symmetric and all its diagonal entries are 1. Most of these entries are described by rephrasing [15, Lemma 1].

Theorem 1. *If $m_{i-1,j-1}$ is the order of $f_{i-1}f_{j-1}$ with $1 < i < j \le n$, then*

1. $m_{i-1,i-1} = 1$,
2. $m_{i-1,j-1} = m_{j-1,i-1}$,
3. $m_{1,2} = 3$, *and*
4. *if $j \ge 4$ then*
   (a) *If $1 < i \le \lfloor \frac{j}{2} \rfloor$,then $m_{i-1,j-1} = 4$.*
   (b) *If $1 < \lfloor \frac{j}{2} \rfloor < i < j-1$, then*

$$m_{i-1,j-1} = \begin{cases} 2q(q+1), & \text{if } r \ge 2, t \ge 2, \text{ or } r = 1, t \ge 2, q \text{ is even,} \\ & \text{or } r \ge 2, t = 1, q \text{ is odd;} \\ q(q+1), & \text{if } r = 1, t \ge 2, q \text{ is odd, or } r \ge 2, t = 1, q \text{ is even,} \\ & \text{or } r = 1, t = 1; \\ 2q, & \text{if } r = 0. \end{cases}$$

   where $d = j - i$, $q = \lfloor \frac{j}{d} \rfloor$, $r = j \pmod{d}$ and $t = d - r$.
   (c) *If $i = j-1$, then $m_{i-1,j-1} = j$*

Order of $f_i f_j f_k$

We now discuss the order $m_{i,j,k}$ of $f_i f_j f_k$. It turns out that all we need is to understand the order in the case $i \le j \le k$, as the order of $f_{\sigma(i)} f_{\sigma(j)} f_{\sigma(k)}$ is also $m_{i,j,k}$, as shown in the following lemma.

Lemma 2. *For all i, j, k with $1 \le i, j, k \le n$ and any permutation σ of $\{i, j, k\}$, the order of $f_i f_j f_k$ is the same as the order of $f_{\sigma(i)} f_{\sigma(j)} f_{\sigma(k)}$.*

Here are a collection of partial results on the orders of elements of the form $f_i f_j f_k$. Specifically these are all of the relations where the leftmost generator is f_1, i.e. all of the orders of $f_1 f_j f_k$.

Theorem 3. *If $m_{1,j-1,k-1}$ is the order of $f_1 f_{j-1} f_{k-1}$ with $1 < j < k \le n$, then*

1. $m_{1,1,j-1} = m_{1,j-1,j-1} = 2$,
2. *if* $j \ge 6$ *then* $m_{1,2,j-1} = 6$,
3. *if* $j = k-1$ *then* $m_{1,j-1,k-1} = k-1$,
4. *if* $j = k-2$ *and k is odd or* $j = k-3$ *and* $2 \ne k \pmod 3$ *then* $m_{1,j-1,k-1} = k$,
5. *if* $k \ge 5$ *then*

(a)
$$m_{1,j-1,k-1} = \begin{cases} 4q, & \text{if } r = 0, d \ge 4; \\ 2q+1, & \text{if } r = 1, d = 2; \\ q(3q+1), & \text{if } r = 1, d = 4 \text{ or } r = 1, d \ge 5, q \text{ is odd}; \\ 2q(3q+1), & \text{if } r = 1, d \ge 5, q \text{ is even}. \end{cases}$$

(b)
$$m_{1,j-1,k-1} = \begin{cases} q(q+1), & \text{if } r = 2, d = 3, \text{ or } r = 2, d \ge 4, q \text{ is odd, or} \\ & r = 3, d = 4, 0 = q \pmod 3, \text{ or } r = 3, d \ge 5, \\ & q = 3 \pmod 6, \text{ or } r \ge 4, d \ge 5, 0 = q \pmod 4; \\ 2q(q+1), & \text{if } r = 2, d \ge 4, q \text{ is even, or } r = 3, d \ge 5, \\ & 0 = q \pmod 6, \text{ or } r \ge 4, d \ge 5, 2 = q \pmod 4; \\ 3q(q+1), & \text{if } r = 3, d = 4, 0 \ne q \pmod 3, \text{ or } r = 3, d \ge 5, \\ & 1, 5 = q \pmod 6; \\ 4q(q+1), & \text{if } r \ge 4, d \ge 5, q \text{ is odd}; \\ 6q(q+1), & \text{if } r = 3, d \ge 5,\ 2, 4 = q \pmod 6; \end{cases}$$

where $d = k - j, q = \left\lfloor \frac{k}{d} \right\rfloor$, *and* $r = k \pmod d$.

In the next section, we describe the pancake matrix for B_n, and make connections to the corresponding pancake graph of B_n.

B_n results

We now provide a complete description for the order of $f_i^B f_j^B, 0 \leq i, j \leq n-1$ for signed permutations.

Theorem 4. *If $m^B_{i-1,j-1}$ is the order of $f^B_{i-1} f^B_{j-1}$ with $1 \leq i < j \leq n$, then*

1. $m^B_{i-1,i-1} = 1$,
2. $m^B_{i-1,j-1} = m^B_{j-1,i-1}$,
3. *If* $1 < i \leq \lfloor \frac{j}{2} \rfloor$ *(with* $j \geq 4$*) then* $m^B_{i-1,j-1} = 4$.
4. *If* $1 \leq \lfloor \frac{j}{2} \rfloor < i < j-1$ *(with* $j \geq 4$*), then*
$$m^B_{i-1,j-1} = \begin{cases} 2q & \text{if } r = 0, \text{ and} \\ 2q(q+1) & \text{if } r \neq 0 \end{cases}$$
where $d = j - i$, $q = \lfloor \frac{j}{d} \rfloor$, $r = j \pmod{d}$,
5. *If* $i = j - 1$ *(with* $j \geq 3$*) then* $m^B_{i-1,j-1} = 2j$.

Connection with the burnt pancake graph

The Pancake graph for S_n, and in particular its cycle structure, has been extensively studied (see, for example, [2, 12, 13, 14, 16, 15, 17]). From the results of Theorem 4, one can derive results regarding the cycle structure of the Cayley graph corresponding to B_n generated by P^B.

Theorem 5. *The Cayley graph of B_n with the generators P^B (burnt pancake graph for B_n), with $n \geq 2$, contains a maximal set of $\frac{2^n n!}{\ell}$ independent ℓ-cycles of the form $(f_i^B f_j^B)^k$, with $0 \leq i < j < n$, $\ell = 2k$ and $k = (M_n^B)_{i+1,j+1}$, the $(i+1, j+1)$ entry in M_n^B.*

A straightforward observation is that any cycle in the burnt pancake graph is even. Indeed, if we notice the action pancake generators have on the first element of any signed permutation, in window notation, we see that its sign changes every time it is multiplied by a pancake generator. Therefore, if there were an odd cycle, $f^B_{i_1} f^B_{i_2} \cdots f^B_{i_{2k+1}}$, then the sign of $f^B_{i_1} f^B_{i_2} \cdots f^B_{i_{2k+1}}(1)$ would be negative. Hence we have the following proposition.

Proposition 6. *If C is a cycle in the burnt pancake graph for B_n, then the length of C is even.*

We recall that a *chord* in a cycle C is an edge not belonging to a C that connects two vertices of C. Just in the case for the pancake graph of S_n (see [15]), the cycles described in Theorem 5 have no chords. We make this formal in the following Lemma.

Lemma 7. *The cycles described in Thoerem 5 have no chords. Moreover, the burnt pancake graph for B_n does not have a simple cycle of length six.*

The authors are grateful to Ivars Peterson for an introduction to the subject whose talk in 2014 at the EPaDel sectional meeting inspired this paper. We also thank Jacob Mooney and Kyle Yohler for independently writing computer code to verify our results.

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Enumerating Permutations Sortable by k Passes Through a Pop-Stack

Anders Claesson University of Iceland

This talk is based on joint work with Bjarki Ágúst Guðmundsson

Introduction

Knuth [2, Exercise 2.2.1.5] noted that permutations sortable by a stack are precisely those that do not contain a subsequence in the same relative order as the permutation 231. This exercise inspired a wide range of research and can be seen as the starting point of the research field we now call permutation patterns.

A closely related problem is that of sorting permutations by k passes through a stack, where the elements on the stack are required to be increasing when read from top to bottom. West [5] characterized the permutations sortable by two passes through a stack in terms of pattern avoidance and conjectured their enumeration, a conjecture that was subsequently proved by Zeilberger [6]. Permutations sortable by three passes have been characterized by Úlfarsson [4], but their enumeration is unknown.

In other variations of Knuth's exercise different data structures are used for sorting. One notable example, introduced by Avis and Newborn [1] in 1981, is that of pop-stacks: a stack where each pop operation completely empties the stack. Let $P_k(x)$ be the generating function for the permutations sortable by k passes through a pop-stack. The generating function $P_2(x)$ was recently given by Pudwell and Smith [3] (the case $k = 1$ being trivial). They characterized the permutations sortable by two passes through a pop-stack in terms of pattern avoidance. They also gave a bijection between certain families of polyominoes and the permutations sortable by one or two passes through a pop-stack, but noted that the bijection does not generalize to three passes.

We show that $P_k(x)$ is rational for any k. Moreover, we give an algorithm to derive $P_k(x)$, and using it we determine the generating functions $P_k(x)$ for $k \leq 6$.

Main results

A single pass of the pop-stack sorting operator formally works as follows. Processing a permutation $\pi = a_1 a_2 \dots a_n$ of $[n] = \{1, \dots, n\}$ from left to right, if the stack is empty or its top element is smaller than the current element a_i then perform a single pop operation (a), emptying the stack and appending those elements to the output permutation; else do nothing (d). Next, push a_i onto the stack and proceed with element a_{i+1}, or if $i = n$ perform one final pop operation (a), again emptying the stack onto the output permutation, and terminate. Define $P(\pi)$ as the final output

7	5	2	4	9	1	8	6	3
2	5	7	4	1	9	3	6	8
2	5	1	4	7	3	9	6	8
2	1	5	4	3	7	6	9	8
1	2	3	4	5	6	7	8	9

Figure 1: A sorting trace ...

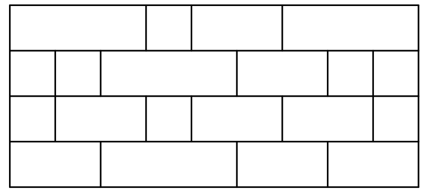


Figure 2: ...and its sorting plan

```
a d d a a d a d d a
a a a d d a d a a a
a a d a a d a d a a
a d a d d a d a d a
```

Figure 3: The same sorting plan ...

0, 9, 10, 5, 5, 10, 5, 10, 9, 0

Figure 4: ...and its encoding

permutation and $w(\pi)$ as the word over the alphabet $\{\texttt{a}, \texttt{d}\}$ defined by the operations performed when processing π. Note that $w(\pi)$ will always begin and end with the letter a. We will call any word of length $n+1$ with letters in $\{\texttt{a}, \texttt{d}\}$ that begin and end with the letter a an *operation sequence*.

Consider applying the pop-stack operator P to a permutation π of $[n]$. Start by interleaving $w(\pi)$ with π. Replacing a with a bar and d with a space, and placing $P(\pi)$ below this string, we get:

7	5	2	4	9	1	8	6	3
2	5	7	4	1	9	3	6	8

We call the numbers between pairs of successive a's *blocks*. Above, the blocks are 752, 4, 91, and 863. Note that $P(\pi)$ can be obtained from π by reversing each block. Further, $w(\pi) = c_1 c_2 \ldots c_{n+1}$ is simply the ascent/descent word of $-\infty\pi\infty$; i.e. $c_1 = c_{n+1} = \texttt{a}$ and, for $2 \leq i \leq n$, $c_i = \texttt{a}$ if $i-1$ is an ascent, and $c_i = \texttt{d}$ if $i-1$ is a descent.

A figure such as the one above can be extended to depict multiple passes. Applying P to the example permutation, π, until it is sorted gives Figure 1. We will call such figures *sorting traces*, or *traces* for short. The structure that remains when removing the numbers from a trace—see Figure 2—we call its *sorting plan*. Each row of a sorting plan corresponds to an operation sequence, and for convenience we shall number the rows 1 through k, from top to bottom. The example sorting plan can be viewed as the array of operation sequences given in Figure 3. By interpreting each column as a binary number with $a = 0$ and $d = 1$ the sorting plan can more compactly be represented, or *encoded*, with a sequence of numbers: see Figure 4.

Our goal is to count the permutations of $[n]$ that are sortable by k passes through a pop-stack, or *k-pop-stack-sortable* permutations for short. Starting with a k-pop-stack-sortable permutation of $[n]$ and performing k passes of the pop-stack sorting operator results in a trace of length n and order k. Conversely, the first row of that trace is the k-pop-stack-sortable permutation we started with. Thus, k-pop-stack-sortable permutations of $[n]$ are in one-to-one correspondence with traces of length n and order k. Those are, in turn, in one-to-one correspondence with their sorting plans, and hence it suffices to count sorting plans of length n and order k, or, equivalently, their encodings.

Let us call any k-tuple of operation sequences of length $n+1$ an *operation array* of length $n+1$ and order k. Note that sorting plans are operation arrays, but not all operation arrays are sorting plans. We shall characterize those operation arrays that are sorting plans, then count the sorting plans of order k, and by extension the k-pop-stack-sortable permutations. For instance, it is easy to see that each operation sequence represents a sorting plan of order 1. An operation sequence starts and ends with the letter a. The remaining letters can be either a or d, and we have rediscovered the well know fact that the number of 1-pop-stack-sortable permutations of $[n]$ is 2^{n-1}.

While this simple example outlines our approach to count k-pop-stack-sortable permutations, it is a little too simple, as for larger k, most operation arrays will not be sorting plans. As an example, we present the following lemma.

Lemma 1. *In a trace of order 2 or greater, each operation sequence—except for the first one—contains at most 2 consecutive* d*'s. Or, equivalently, each row of the sorting plan—except for the first one—has blocks of size at most 3.*

This lemma characterizes a certain class of operation arrays that are not sorting plans, namely those that have at least one block of size 4 or larger in rows 2 to k. A more general approach towards characterizing operation arrays that are not sorting plans is to consider what we call *forbidden segments*. The definition of a segment is as follows: Let ψ be the bijection mapping a sorting plan to its encoding. Let M be a sorting plan of length n and suppose that $1 \leq i \leq j \leq n+1$. Let $\psi(M) = c_1 c_2 \dots c_{n+1}$ be the encoding of M. Then we call $S = \psi^{-1}(c_i c_{i+1} \dots c_j)$ a *segment* of M.

We shall not formally define what the forbidden segments are, but they are certain segments that must be avoided in order for an operation array to be a sorting plan. Figures 5, 6, 7 and 8 are meant to convey the rough idea, in which a semitrace is a simple generalization of a trace: Figure 6 shows how the pair of numbers $(2,4)$ progress through the semitrace. The segment $T_{2,4}$, shown in Figure 7, is the smallest segment that contains all blocks on rows 2, 3, 4, and 5 with at least one circled element.

Any operation array of order 5 that contains this segment, no matter where it occurs horizontally, will fail to be a sorting plan/trace, not just as the above semitrace. This is because we can follow the two elements a and b playing the roles of 2 and 4 from the bottom to the second row, as shown in Figure 8.

Definition 2. A segment of order k is *bounded* if each of its blocks in rows 2 to k has size at most 3.

We establish the following lemmas.

Lemma 3. *An operation array is a sorting plan if and only if it does not contain any bounded forbidden segment and each block on rows 2 through k is of size at most 3.*

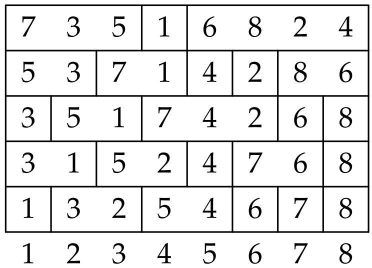


Figure 5: An example semitrace

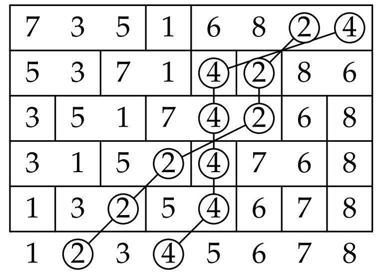


Figure 6: The progress of 2 and 4

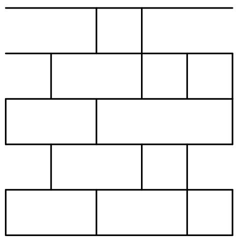


Figure 7: The segment $T_{2,4}$

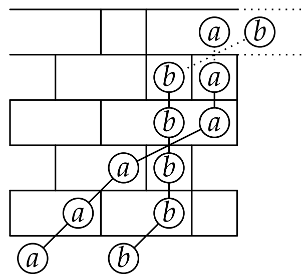


Figure 8: Following a and b

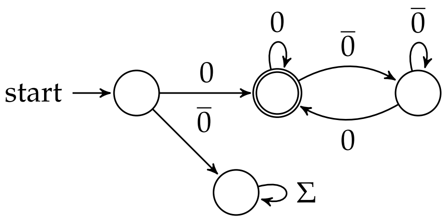


Figure 9

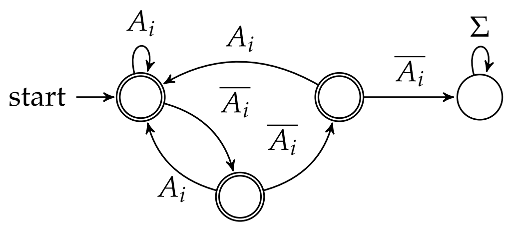


Figure 10

Lemma 4. *Let T be a semitrace of length n and order k and let a and b be two distinct elements of $[n]$. If $T_{a,b}$ is a bounded segment, and there is a block, not on the first row, that includes both a and b, then $|T_{a,b}| \leq 4k - 5$.*

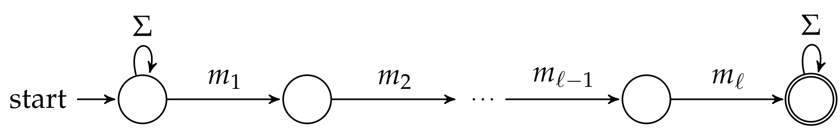
Lemma 5. *For a fixed k, there are finitely many bounded forbidden segments of order k, and they can be listed.*

Recall that we can encode an operation array of length n and order k as a sequence of n integers, each in the range $[0, 2^k - 1]$. In this way we can consider operation arrays as strings of a formal language over the alphabet $\Sigma = \{0, 1, \ldots, 2^k - 1\}$. Conversely, strings over this alphabet can be considered as operation arrays, under one condition: that they both begin and end with a solid boundary. Noting that a solid boundary corresponds to the integer 0 from Σ, and letting $\overline{0} = \Sigma \setminus \{0\}$, the DFA W in Figure 9 recognizes the strings over Σ that begin and end with a solid boundary, i.e. the strings that correspond to an operation array.

Recall from Lemma 3 that an operation array is a sorting plan if and only if it does not contain any bounded forbidden segments and each block on rows 2 through k is of size at most 3. We shall start with the latter condition.

If A_i is the set of symbols from Σ that represent a column from the operation array that has a bar in the ith row, and $\overline{A_i} = \Sigma \setminus A_i$, then the intersection of W with the automaton, R_i, in Figure 10 recognizes the operation arrays that have blocks of size at most 3 in row i. Therefore, the set of operation arrays that have blocks of size at most 3 in all but the first row is recognized by the automaton $W \cap R_2 \cap \cdots \cap R_k$.

The other condition that sorting plans satisfy is that they do not contain any bounded forbidden segments. Consider a segment M and let us encode it in the same manner as we encode operation arrays, resulting in the sequence $m_1, \ldots, m_\ell$. Note that an operation array A contains M if and only if the encoding of A contains $m_1 \cdots m_\ell$ as a factor. Furthermore, the following nondeterministic finite automaton, Q_M, recognizes the set of strings over Σ that contain the encoding of M as a factor:



Taking the complement of Q_M we get an automaton $\overline{Q_M}$ that recognizes the set of strings over Σ that do not contain the factor M. In particular, if F is a forbidden segment, then $W \cap \overline{Q_F}$ recognizes the set of operation arrays that do not contain F. Let $\mathcal{F}$ be the set of bounded forbidden segments, which is finite by Lemma 5. Then

$$S = W \cap \bigcap_{i=2}^{k} R_i \cap \bigcap_{F \in \mathcal{F}} \overline{Q_F}$$

recognizes the set of operation arrays that have blocks of size at most 3 in rows 2 through k, and do not contain any bounded forbidden segments. Hence, by Lemma 3, S recognizes exactly the set of sorting plans, giving us:

Proposition 6. *The language $S = \{ w \in \Sigma^* \mid w$ is encoding a sorting plan $\}$ is regular.*

Theorem 7. *For a fixed k, the generating function $P_k(x) = \sum_{n=0}^{\infty} p_n x^n$, where p_n is the number of k-pop-stack-sortable permutations of length n, is rational.*

All of the above results are constructive, meaning that the generating function can be computed for any fixed k. We did so for $k = 1, \ldots, 6$. In Table 1 we list the resulting generating functions for $k \in \{1, 2, 3\}$. For $k \in \{4, 5, 6\}$ the expressions are too large to display here, but all the generating functions, source code, and text files defining the DFAs can be found on GitHub at

`https://github.com/SuprDewd/popstacks`

k	Generating function
1	$\dfrac{x-1}{2x-1}$
2	$\dfrac{x^3+x^2+x-1}{2x^3+x^2+2x-1}$
3	$\dfrac{2x^{10}+4x^9+2x^8+5x^7+11x^6+8x^5+6x^4+6x^3+2x^2+x-1}{4x^{10}+8x^9+4x^8+10x^7+22x^6+16x^5+8x^4+6x^3+2x^2+2x-1}$

Table 1: The generating functions for the k-pop-stack-sortable permutations, $k \leq 3$

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Pattern Avoidance in Motzkin Paths

Daniel Daly Southeast Missouri State University

This talk is based on joint work with Mary Ramey

In [1], Bernini, Ferrari, Pinzani and West began the study of pattern avoidance for lattice paths and specifically obtained many enumerative results concerning pattern avoidance in Dyck paths. Here we study pattern avoidance for Motzkin paths and obtain enumeration results for patterns of small length.

A *Motzkin path* is a lattice path from $(0,0)$ to $(n,0)$ using only the steps $(1,1)$, $(1,-1)$ and $(1,0)$ and never going below the x-axis. Hence any Motzkin path from $(0,0)$ to $(n,0)$ can be encoded as a sequence of n letters on the alphabet $\{U, D, H\}$ where the number of U's equals the number of D's and the number of U's to the left of any point is greater than or equal to the number of D's to the left of that same point.

A Motzkin path π *contains* another path σ if π contains σ as a subsequence. If not π *avoids* σ. We define $M_n(\sigma)$ to be the set of all Motzkin paths avoiding σ and $m_n(\sigma)$ to be the cardinality of $M_n(\sigma)$. As might be expected, the enumeration of $m_n(\sigma)$ for various choices of σ involves Catalan numbers, central binomial coefficients, powers of 2 and other classical combinatorial sequences.

For paths of length 3, there are two Wilf-equivalence classes: $\{UDH, HUD\}$ and $\{UHD\}$. For these, we have

Theorem 1.

$$m_n(UDH) = m_n(HUD) = \binom{n}{\lfloor n/2 \rfloor} \quad (A001405)$$

Theorem 2.

$$m_n(UHD) = 1 + \sum_{k=1}^{\lfloor n/2 \rfloor} (n - 2k + 1)C_k$$

To prove Theorem 1, we create a bijection between $M_n(UDH)$ and a set of left Dyck factors. There is also a nice recurrence for this sequence which generates an interesting combinatorial identity involving Catalan numbers and central binomial coefficients.

In this talk, we will expand on these results and also discuss enumeration results for patterns of length four and some patterns of more general length. Some enumeration sequences found are well known in the OEIS, though many have been found that appear to be new.

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UNIVERSAL PERMUTATIONS

Michael Engen University of Florida

This talk is based on joint work with Michael Albert, Jay Pantone, and Vince Vatter

The problem of finding the length of a shortest permutation containing all permutations of length n appears to have first been considered by Arratia [2] in 1999. He presented the trivial upper and lower bounds of n^2 and n^2/e^2, respectively. We say a permutation is n-universal if it contains all permutations of length n (the alternate term *superpattern* is sometimes used in the literature).

To-date, no progress has been made on the lower bound of n^2/e^2 in this case, but there have been two previous improvements to the upper bound. In a paper first appearing in 2007 but based on 2002 work, Eriksson, Eriksson, Linusson, and Wästlund [3] improved the upper bound to $2n^2/3$ (up to a little-oh factor). In 2009, Miller [5] further improved this bound to $\binom{n+1}{2}$.

We establish the following.

Theorem 1 (Engen and Vatter, in preparation)**.** *There is an n-universal permutation of length* $\lceil (n^2+1)/2 \rceil$.

Given a permutation class $\mathcal{C}$, the permutation π is said to be *n-universal for* $\mathcal{C}$ if π contains all of the permutations of length n in $\mathcal{C}$. In [1], we give an explicit formula for the length of the shortest permutations which are n-universal for the class of layered permutations. Moreover, we show that there are shortest permutations which are n-universal for the class of layered permutations which are themselves layered, proving a conjecture of Gray [4].

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Automatic Enumeration of Grid Classes

Unnar Freyr Erlendsson Reykjavik University

This talk is based on joint work with Christian Bean and Henning Úlfarsson

Given a matrix $\mathcal{M}$ whose entries are permutation classes, the permutations in the *grid class* defined by $\mathcal{M}$, $\text{Grid}(\mathcal{M})$, are those which can have a grid drawn on it such that the subpermutation in each box is in the corresponding permutation class in $\mathcal{M}$. Figure 1 shows an example of this.

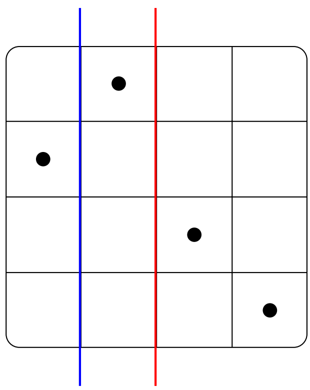

Figure 1: The red and the blue lines show two different griddings of 3421 with $\mathcal{M} = \begin{pmatrix} \text{Av}(21) & \text{Av}(12) \end{pmatrix}$.

In this paper, we are interested in $1 \times N$ grid classes, and we will write $\text{Grid}(A|B)$ to represent the grid class defined by the 1×2 matrix $\mathcal{M} = \begin{pmatrix} \text{Av}(A) & \text{Av}(B) \end{pmatrix}$.

In [1] they enumerated 1×2 grid classes of the form $\text{Grid}(\sigma, \tau)$ where $\sigma \in \mathcal{S}_3, \tau \in \mathcal{S}_2$. To do this they introduced the notion of greedily gridding the permutation by putting as many of the points in the first box as possible. We extend this idea to general $1 \times N$ grid classes and using the TileScope algorithm automatically enumerate some of these.

The TileScope algorithm can be used to try to enumerate the number of griddings on a slightly more fine grained version of grid classes called *tilings*. In the definition of grid classes you start with a permutation and try to draw on a grid but in tilings you go in the other direction, start with a grid and draw the permutation. A *gridded permutation* is a permutation together with integer cells in each points exponent, called the *gridded position*, that tells you which cell the point is in. We say a gridded permutation is local in cell (i, j) if the gridded position of all points is (i, j) and denote it as $\pi^{(i,j)}$. We define containment of gridded permutations in the same way as permutations but adding the condition that the cells of the pattern and the occurence match. Figure 2 gives an example of containment of gridded permutations.

A *tiling* is a triple $\mathcal{T} = ((n, m), \mathcal{O}, \mathcal{R} = \{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k\})$, where $\mathcal{O}$ is a set of gridded permutations called *obstructions* to be avoided and $\mathcal{R}$ is a set of sets of gridded permu-

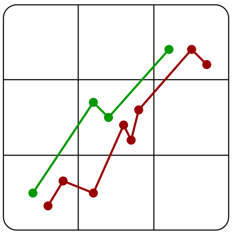

Figure 2: The red gridded permutation $1^{(1,1)}3^{(1,1)}2^{(1,2)}5^{(2,2)}4^{(2,2)}6^{(2,2)}8^{(3,3)}7^{(3,3)}$ contains the green gridded permutation $1^{(1,1)}3^{(2,2)}2^{(2,2)}4^{(3,3)}$ four times, $1^{(1,1)}5^{(2,2)}4^{(2,2)}8^{(3,3)}$, $1^{(1,1)}5^{(2,2)}4^{(2,2)}7^{(3,3)}$, $3^{(1,1)}5^{(2,2)}4^{(2,2)}8^{(3,3)}$ and $3^{(1,1)}5^{(2,2)}4^{(2,2)}7^{(3,3)}$.

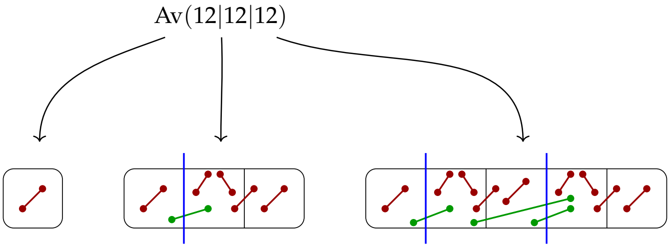

Figure 3: Disambiguation of Av(12|12|12). Red points connected by lines represent obstructions while green points connected by lines represent requirements. Note that in the bottom right tiling the two requirements going to cell $(4,1)$ are in the same set of requirements, that is you only have to contain one of them, not both. The blue lines represents the drawing of the grid.

tations to be contained called *requirements*. Then Grid$(\mathcal{T})$ is the gridded permutations with gridded positions in $\{1,\ldots,n\} \times \{1,\ldots,m\}$ that avoid all σ in $\mathcal{O}$ and contain all $\mathcal{R}_1$ in $\mathcal{R}$. We will say $\mathcal{T} = \text{Grid}(\mathcal{T})$, and refer to gridded permutations being on a tiling.

Disambiguation of a grid class

Theorem 1. *Let $\mathcal{G}$ be a $1 \times N$ grid class. There exists a disjoint union of tilings $\mathcal{T}$ such that the gridded permutations in $\mathcal{T}$ are in bijection with the griddable permutations in $\mathcal{G}$.*

We start with a $1 \times N$ grid class $\mathcal{G}$ where each cell $(1,i)$ contains a permutation class with a finite basis, B_i. Define $\mathcal{O}_i = \{\pi^{(1,i)} : \pi \in B_i\}$. We build the set of tilings from left to right by essentially greedily gridding a permutation $\pi \in \mathcal{G}$. Consider the first cell, either π can be fully contained in cell $(1,1)$ or it contains one of the underlying permutations of an obstruction in $\mathcal{O}_1$ meaning it will have to stretch to the next cell. We will represent this as a disjoint union of tilings where the first tiling, t_1, only

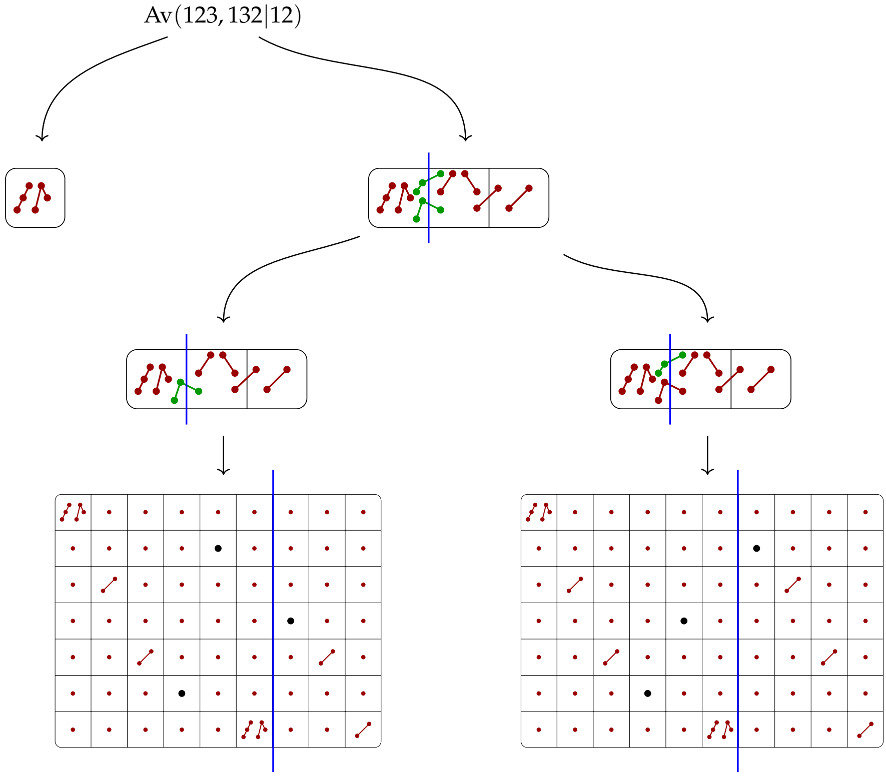

Figure 4: Combinatorial specification for Av(123, 132|12) where the black points represent precisely a point, red are obstructions and green are requirements. Note that in the right child tiling of the root the requirements are in the same set of requirements. The blue lines represent the grid being drawn. From this tree we can find that the generating function for this class is $\frac{3x^4-15x^3+17x^2-7x+1}{12x^4-28x^3+23x^2-8x+1}$.

contains a single cell with the obstructions from $\mathcal{O}_1$.

To build t_i where $i > 1$, we carry over all of the obstructions and requirements from the tiling t_{i-1} and we will extend it by adding two new cells, $(1, 2(i-1))$ and $(1, 2i-1)$. If π can not be gridded on any of t_k, $k < i$ then it must not have been fully contained in the first $i-1$ cells of $\mathcal{G}$. We will create a set of requirements, $\mathcal{R}_i$, which will represent the idea that π could not be gridded on any of the previous tilings and therefore must stretch into cell $(1, i)$ in $\mathcal{G}$. For every obstruction σ which contains at least a single point in cell $(1, 2(i-1)-1)$ in t_i we will create a requirement τ with the same underlying permutation as σ but the gridded position of the last point will be in cell $(1, 2(i-1))$. We will add the obstructions $12^{(2i-1,1)}$ and $21^{(1,2i-1)}$ to ensure that cell $(1, 2i-1)$ contains only a single point. By doing that we are ensuring that it is the leftmost point in cell $(1, i)$ in $\mathcal{G}$.

We also need to make sure that we restrict these two cells only to the permutations that can be gridded on cell $(1, i)$ in $\mathcal{G}$. For every obstruction σ in cell $(1, i)$ of $\mathcal{G}$ we will add two obstructions, the obstruction where σ is local in cell $(1, 2i-1)$ and the

$n\|m$	All	Success
1|1	15	5
1|2	25	11
1|3	24	12
1|4	11	5
1|5	2	1
2|1	19	1
2|2	49	7
2|3	52	30
2|4	24	18
2|5	4	3
3|1	19	2
3|2	52	29
3|3	64	54
3|4	32	31
3|5	6	6
4|1	9	1
4|2	24	19
4|3	34	34
4|4	20	20
4|5	4	4
5|1	2	0
5|2	5	4
5|3	7	7
5|4	5	5
5|5	1	1
Total	509	310

Table 1: Table showing the number of successes for $\text{Av}(A|B)$. $n|m$ refers to the number of patterns on each side.

gridded permutation with the same underlying permutation as σ where the gridded position of the first point is in cell $(1, 2(i-1))$ and the rest are in cell $(1, 2i-1)$. Now $t_1, \dots, t_n$ form a disjoint of tilings as per Theorem 1. We call this method the *disambiguation* of a grid class. See Figure 3 for the disambiguation of $\text{Av}(12|12|12)$.

Using disambiguation we can create a disjoint union of tilings which allows us to use the TileScope algorithm to find a combinatorial specification and therefore also finding the enumeration for some grid classes automatically.

Results

Using the method of disambiguating grid classes in combination with the TileScope algorithm we have started enumerating all grid classes of the form $\text{Av}\, A|B$ where A and B are subsets of $\mathcal{S}_2 \cup \mathcal{S}_3$. We only considered grid classes which are lexicographically minimum down to symmetry and where at least one side is not a finite class, in total there are 1100 grid classes we considered. Table 1 shows the successes we had running the TileScope algorithm for 72 hours on the disambiguated grid classes where a success means that we have found a combinatorial specification for the given grid class which gives the enumeration. Figure 4 shows one of the combinatorial specifications found by TileScope.

In particular, TileScope was able to enumerate 4 of the 6 juxtapositions discussed in [1], namely $\mathrm{Av}(213|21), \mathrm{Av}(132|12), \mathrm{Av}(213|12)$ and $\mathrm{Av}(132|21)$.

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Knots and Permutations

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This talk is based on joint work with Joel Hass, Nati Linial, and Tahl Nowik

The study of permutations as representing knots was introduced by C. Adams et al. [2], followed by a sequence of recent works [1, 3, 12, 11, 13, 10]. In this proposed talk, we'll describe this connection between permutations and knots, and how questions about properties of knots relate to ones about permutation statistics.

In particular, certain knot invariants can be expressed in terms of the number of occurrences of permutation patterns. For example, the framing number of a knot yields a circular variant of the inversion number of a permutation.

It is natural to define a model for random knots based on this representation, with the uniform distribution on permutations. Such knot invariants can be studied as permutation statistics in terms of their moments, limit distributions, etc. Our analysis yields the first such results for any representation of knots.

The Construction

Intuitively, a *knot* is a simple closed curve, embedded in the three-dimensional space, like a rope whose two ends are joined together. As usual for topological objects, two knots are equivalent if one can be deformed into the other by continuous moves, where stretching and squeezing are allowed but no cutting and pasting. See formal definitions and further details in [4, 16].

There are infinitely many different knots, and it is not an easy task to sort out and describe them all. A central theme in knot theory is to esablish and study convenient discrete descriptions for all knots, using planar diagrams, closed braids, polygonal paths on the grid, and more.

We now describe [2]'s basic construction of a knot $K(\pi)$, given a permutation π of odd size n. We consider a curve in $\mathbb{R}^3$ whose projection to the plane has one multi-crossing point of n straight arcs, connected by petals as in Figure 1. The arcs' heights as they pass above the center are determined by the permutation π, which uniquely defines the knot. Specifically, the entries of π list the heights of the n arcs according to their order of occurrence as we traverse the curve, starting from an arbitrary petal.

Theorem 1 (Adams et al. [2]). *Every knot K is obtained as $K(\pi)$ by the above construction from some permutation π.*

This means that one can use permutations in order to represent all knots. It is natural to ask how efficient this representation is, compared to other ones. A planar *knot diagram* represents a knot by its projection to the plane, having a finite number of

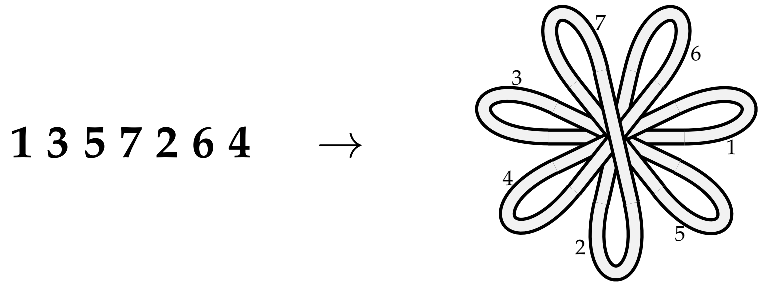


Figure 1: The heights of the arcs in the petal diagram are given by the permutation.

double points, called *crossings*, and marked to distinguish over- and under-strands. For example,  is one diagram of the *trefoil* knot.

By the following theorem, the efficiency of permutations in representing all knots is quite comparable to that of diagrams.

Theorem 2 (Even-Zohar et al. [13]). *Every nontrivial knot K, given by a diagram with c crossings, is obtained as $K(\pi)$ for some permutation π of size at most $2c - 1$.*

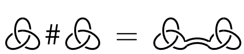
For example, the 3-crossing trefoil knot, is $K(24135)$. The *figure-eight* knot, usually represented by a diagram with 4 crossing, is also given by the permutation 1357264 as in Figure 1, and so on.

Some Properties

We next examine some simple operations on permutations, and their effect on the corresponding knots. Let $\rho \in S_n$ be the rotation $\rho(x) = x + 1 \bmod n$. Note that the above construction is invariant to rotations from both directions. Indeed, we are free to move the starting point to the next petal, or move the upper strand around everything to be the lower one. In conclusion,

Proposition 3. *For every odd n and $\pi \in S_n$, $K(\pi) = K(\pi \circ \rho) = K(\rho \circ \pi)$*

It follows that one can assume without loss of generality that $\pi(n) = n$. For such $\pi \in S_n$ denote by $\pi' \in S_{n-1}$ the permutation obtained by omitting the last entry.

The operation of *connected sum*,  gives some structure to the set of knots. Indeed, a theorem by Schubert states that every knot can be uniquely decomposed as a connected sum of *prime* knots, which are knot that cannot be decomposed further. Given two knots represented as permutations, the direct sum of these permutations yields the connected sum.

Proposition 4. *If $\pi \in S_n$ and $\sigma \in S_m$ for odd n and m, then $K(\pi)\#K(\sigma) = K(\pi' \oplus \sigma)$.*

For example, the connected sum depicted above, of two trefoil knots, also known as the *granny* knot, is $K(24135) \# K(24135) = K(2413 \oplus 24135) = K(241368579)$.

Counting Unknots

It follows from the above that the representation of knots by permutations, similar to other common methods, is far from being one-to-one. It is interesting to understand how many different knots are obtained from permutations of size n, and how many permutations represent a given knot K.

By taking $\pi \in S_n$ uniformly at random, these become questions about the distribution of a random knot $K(\pi)$. The study of random knots is motivated by theoric as well as practical perspectives, and this particular random model, which we call *Petaluma*, seems to have certain desirable features. See [10] for a survey.

The Delbruck–Frisch–Wasserman conjecture [7, 14], states that a typical random knot sould be non-trivial, i.e., not equivalent to the plain circle $\bigcirc$, the *unknot*. This conjecture is known to hold in several random knot models [19, 17, 9, 8, 5], as well as the one discussed here.

Theorem 5 (Even-Zohar et al. [13]). *The number of permutations $\pi \in S_n$ such that $K(\pi) = \bigcirc$ is $O(n!/n^{0.1})$.*

The proof of Theorem 5 makes use of the connection between permutation patterns and knot invariants, specifically the Casson invariant discussed below.

A lower bound on the number of unknots, relies on the following *cancellation* move. Suppose that two consecutive entries in $\pi \in S_n$ have consecutive values, such as 45 in 2614537. Then the two corresponding arcs can be pulled away and retract to one petal, without crossing any other part of the knot, as in Figure 2. The remaining heights, 26137 in this example, can be adjusted to produce a permutation in S_{n-2} that represents the same knot. In our example, $K(2614537) = K(24135)$. We remark that here the adjacency of entries and values in $\pi : \mathbb{Z}_n \to \mathbb{Z}_n$ is naturally cyclic, so that 1 and n work too.

Now one can easily construct $\lceil n/2 \rceil!$ different unknotted permutations, just by writing the pairs $\{1, 23, 45, 67, 89, \dots\}$ in any possible order. One can also write 32 instead of 23, and so on. These arguments yields the following lower bound.

Proposition 6. *The number of permutations $\pi \in S_n$ such that $K(\pi) = \bigcirc$, is $\Omega(\sqrt{n!})$.*

This construction only contains some of the permutations that can be reduced to 1 by a sequence of cancellations. Here is a more complicated example with nested pairs: $5273461 \to 52761 \to 32541 \to 321 \to 1$. This naturally raises a new intriguing enumeration problem.

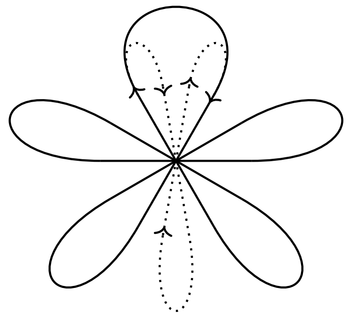


Figure 2: The cancellation move: two adjacent arcs with adjacent heights can be removed.

Question 7. *How many permutations in S_n can be reduced to* 1 *by iterated cancellation of pairs of adjacent entries with adjacent values?*

An answer to Question 7 may improve the lower bound in Proposition 6 at most by an exponential factor. There are however many permutations that represent the unknot, and cannot be eliminated by cancellations. The smallest example is $1, 9, 3, 5, 7, 10, 2, 4, 8, 11, 6$ given in [2].

It remains a central challenge to narrow the gap between the two bounds on the number of permutations in S_n that represent the unknot.

Patterns and Invariants

Invariants are functions that are well-defined on the set of all knots, where usually one has to show that they don't differ on equivalent representations of the same knot. Invariants can be used to distinguish one knot from another, but more generally they can be viewed as tools to classify knots and understand their properties.

Finite type invariants are an important class of knot invariants, that includes the coefficients of the Alexander–Conway polynomial, the modified Jones polynomial, and the Kontsevich integral. It is conjectured that knots can be fully classified by finite type invariants. We omit the general definition, and refer to [6].

Goussarov, Polyak and Viro showed that finite type invariants can be computed from knot diagrams using Gauss diagram formulas [18, 15]. These formulas involve summation over sets of k crossings in a knot diagram, where k is the *order* of the invariant.

Using our representation of knots by permutations, every knot invariant $v(\cdot)$ induces a permutation statistic $v(K(\cdot))$, which is interesting to study. In particular, a formula for a finite type invariant of order k translates into summation over sets of up to $2k$ entries in the permutation. These sums can be viewed as signed or weighted counts of patterns. We discuss two examples for such invariants, of orders one and two.

Signed and Circular Inversion Numbers

Consider the following permutation statistic, which is an alternating version of the inversion number.

$$i(\sigma) \;=\; \sum_{1\le x<y\le n} (-1)^{x+y} \begin{cases} +1 & \sigma(x) < \sigma(y) \\ -1 & \sigma(x) > \sigma(y) \end{cases} \qquad \sigma \in S_n,\ n \text{ odd}$$

By a simple transformation $\pi(x) = \sigma(2x \bmod n)$, one can show that i is equidistributed with another permutation statistic, the *writhe*,

$$w(\pi) \;=\; \sum_{x=1}^{n} \sum_{d=1}^{\lfloor n/2 \rfloor} \begin{cases} +1 & \pi(x) < \pi(x+d \bmod n) \\ -1 & \pi(x) > \pi(x+d \bmod n) \end{cases} \qquad \pi \in S_n,\ n \text{ odd}$$

This permutation statistic essentially counts inversions in π between pairs at distance of up to half-way around the cycle $\mathbb{Z}_n$.

The construction by Adams can be slightly generalized so that π represents a *framed* knot. Then the writhe $w(\pi)$ is exactly the *framing number*, also known as *self-linking*. One can think of a framed knot as a narrow ribbon, and of the framing as counting its "twists", where the framing is zero if the ribbon extends to an embedded surface bounded by the knot. See our paper [11], for more details on framed knots and inversion numbers. The framing number is a first order invariant, and as such it can be computed by summation over pairs.

We study the writhe distribution for random knots, or permutations. We show that its typical order of magnitude is n, unlike $n^{3/2}$ for the usual inversion number. We give explicit formulas for its moments, and deduce a non-normal limit law. See [11] for more details on the limit distribution W.

Theorem 8 (Even-Zohar [11])**.** *There exists a continuous distribution W on $\mathbb{R}$ such that for a uniformly random permutation $\pi \in S_n$,*

$$\frac{w(\pi)}{n} \xrightarrow[n\to\infty]{D} W$$

See [12] for a similar analysis of a classical first-order finite type invariants of 2-component links, the Gauss linking number.

The Casson Invariant

The Casson invariant c_2 is the coefficient of x^2 in the Alexander–Conway polynomial, and the only finite type knot invariant of order two. Here is its definition as a permutation statistic.

$$c_2(\pi) \;=\; \sum_{a\le b\le c\le d\ \star} (-1)^{a+b+c+d+1} \begin{cases} 1 & \pi(a) < \pi(c),\ \pi(b) > \pi(d) \\ 0 & \text{otherwise} \end{cases} \qquad \pi \in S_n,\ n \text{ odd}$$

where a, b, c, d are between 1 and n, and if there is one equality among them then they all have to agree mod 2, except for d if $a = b$, or a if $c = d$. In conclusion, we have a weighted pattern count of size up to four, where the weights only depend on the pattern and on the parities of the indices.

In [12], we study this permutation statistic. We show that it is typically of order of magnitude n^2, and after normalization it tends to be positive, with an intriguing asymmetric distribution. Our main open problem from [12] is to show that the typical order of magnitude of a finite type invariant of order k, is n^k. This is clearly not the case for any permutation statistic that sums over $2k$-long patterns in $\pi \in S_n$. Thus a solution should somehow rely on the knot-theoretic properties of these statistics.

We expect the relation between finite type invariants of knots and permutation patterns to produce more interesting results and connections. Another direction is establishing deterministic, extremal results for these invariants, perhaps in terms the number of petals. That may tell us more about these knot invariants and lead to interesting constructions of knots.

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Stack Sorting with Increasing and Decreasing Stacks

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This talk is based on joint work with Giulio Cerbai

The problem of sorting a permutation using a stack was first introduced by Knuth [4] in the 1960s; in its classical formulation, the aim is to sort a permutation using a first-in/last-out device. As it is well known, in this case the set of sortable permutations is a class, whose basis consists of the single element 231, and whose enumeration is given by Catalan numbers.

More generally (see [6]), one can consider a network of sorting devices, each of which is represented as a node in a directed graph; when there is an arc from node S to node T the machine is allowed to pop an element from S and push it into T; if we mark two distinct vertices as the input and the output machine, then the sorting problem consists of looking for a sequence of operations that allows us to move a permutation from the input to the output machine, thus obtaining the identity permutation.

In this framework, some of the typical problems are the following:

- characterize the permutations that can be sorted by a given network;
- enumerate sortable permutations with respect to their length;
- if the network is too complex, find a specific algorithm that sorts "many" input permutations and characterize such permutations.

Concerning the last stated problem, note that, for a given network of devices, although the set of sortable permutations forms a class in general, this is not anymore true if one choose a specific sorting strategy; this approach leads in general to more complicated characterizations which involve other kinds of patterns (as it happens, for instance, for the 2-West stack-sortable permutations [7]).

Although it's very hard to obtain interesting results for large networks, a lot of work has been done for some particular, small networks (see [2] for a dated survey, or [3] for a more recent one); in this work we restrict our attention to the case of stacks in series, with the restriction that the elements are maintained inside each stack either in increasing or in decreasing order. Our starting point is [5], where Rebecca Smith proves that the permutations sorted by a decreasing stack followed by an increasing one form the class $Av(3241, 3142)$.

Many decreasing stacks followed by an increasing one. Generalizing the approach of [5], we will consider here a sorting device made by k decreasing stacks in series, denoted by $D_1, \ldots, D_k$, followed by an increasing stack I. Recall that "decreasing" (resp., “increasing") stack means that the elements inside the stack have to be in decreasing (resp., increasing) order from top to bottom. When $k = 0$, we just have a single increasing stack, so we obtain the usual Stacksort procedure. When $k = 1$, we obtain exactly the DI machine described in [5]. In the sequel we denote our machine with $D^k I$.

The possible operations the $D^k I$ machine can perform are the following:

- d_0: push the next element of the input permutation into the first decreasing stack D_1;
- d_i, for $i = 1, \dots, k-1$: pop an element from D_i and push it into the next decreasing stack D_{i+1};
- d_k: pop an element from D_k and push it into the increasing stack I;
- d_{k+1}: pop an element from the increasing stack I and output it (by placing it on the right of the list of elements that have already been output).

Notice that each operation can be performed only if it does not violate the restrictions of the stacks; in this case, we call it a *legal* operation. For the special case of the operation d_{k+1}, we will assume that d_{k+1} is legal either if we are pushing into the output the smallest among the elements not already in the output or if all the other operations are not legal.

For any given k, we are now interested in characterizing the set

$$B(k) = \{\pi \in S \mid \text{ there is a sequence of legal operations } d_{i_1}, \dots, d_{i_s} \text{ that sorts } \pi\}.$$

If $\pi \in B(k)$, we say that π is k-sortable. Using a standard argument it is easy to show that $B(k)$ is a class, for every k.

The natural way to describe the class $B(k)$ is to understand its basis. Here we show that, when $k = 2$, the basis of $B(k)$ is infinite, by explicitly finding an infinite antichain of permutations which are not 2-sortable and are minimal with respect to the pattern ordering. The construction of the infinite antichain described in the next theorem can be easily adapted to every $k \geq 2$. An extremely useful tool to find such an antichain has been the software *PermLab* [1], developed by Michael Albert.

Theorem 1. *For $j \geq 0$, define the permutation:*

$$\alpha_j = 2j+4, 3, a_1, b_1, a_2, b_2, \dots, a_j, b_j, 1, 5, 2$$

where:

$$\begin{cases} A_j = (a_1, \dots, a_j) = (2j+2, 2j, 2j-2, \dots, 6, 4), \\ B_j = (b_1, \dots, b_j) = (2j+5, 2j+3, 2j+1, \dots, 9, 7). \end{cases}$$

Then the set of permutations $\{\alpha_j\}_{j\geq 0}$ constitutes an infinite antichain each of whose elements is not 2-sortable. Moreover, α_j is minimal with respect to such a property, i.e. if we remove any element of α_j we obtain a 2-sortable permutation. As a consequence, the basis of $B(2)$ is infinite, since it contains the infinite antichain $\{\alpha_j\}_{j\geq 0}$.

A left-greedy algorithm. Instead of making an unrestricted use of the $D^k I$ machine, we may define a specific algorithm, by choosing the priority of each operation. Depending on our choices, we obtain different sorting procedures: each of them determines a different set of sortable permutations, which is interesting to understand.

Our first proposal is a left-greedy procedure, where, at each step, we perform the legal operation d_j having maximum index j. Setting

$$B_{lg}(k) = \{\pi : \pi \text{ is sorted by the left-greedy procedure}\},$$

it turns out that $B_{lg}(k)$ is in fact a class, and we are able to completely characterize it.

Proposition 2. *For every $k \geq 1$, $B_{lg}(k) = Av(231)$.*

Thus, using this left greedy algorithm, we obtain a procedure that sorts precisely the same permutations as Stacksort does. Thus, in a sense, adding any number of decreasing stacks before an increasing one does not improve the sorting power of the machine, if we always perform the leftmost legal operation. This does not mean, however, that this "left-greedy $D^k I$ machine" is equivalent to Stacksort. Indeed, taking for instance $k = 1$ and the input permutation 2341, the left-greedy $D^k I$ machine returns 2134 as output, whereas Stacksort returns 2314. This is of course due to the fact that the elements of the input permutation exit the (last) decreasing stack in a different order in the two procedures. We can therefore define the map $\phi_k : S_n \to S_n$, for $k \geq 1$, that associates to an input permutation π of length n the output of the last stack D_k in the left-greedy algorithm. As a consequence of the last proposition, for each π and $k \geq 1$, $\pi \in Av(231)$ if and only if $\phi_k(\pi) \in Av(231)$. In order to have a better understanding of the left-greedy $D^k I$ machine, it would be interesting to explore more deeply the properties of the maps ϕ_k.

Example. We report the values of $\phi_k(\pi)$ when $\pi = 36257418$ and $k = 1, 2, 3, 4, 5$. It is easy to observe that, for sufficiently large values of k, the succession $\{\phi_k(\pi)\}_{n\in\mathbb{N}}$ eventually becomes constant. However, we do not know precisely when this happens.

$$\begin{aligned} k = 1 &: \quad 36275418, \\ k = 2 &: \quad 37652841, \\ k = 3 &: \quad 37652841, \\ k = 4 &: \quad 38765241, \\ k = 5 &: \quad 38765241. \end{aligned}$$

An almost left-greedy algorithm. There is a better way to design an almost left-greedy algorithm, which is able to sort more permutations. The idea is to give the increasing stack a privileged role, using it only when it is strictly necessary. Formally, at each step we choose to perform the first legal operation according to the following priority rule:

$$d_{k+1} > d_{k-1} > d_{k-2} > \cdots > d_1 > d_0 > d_k,$$

where $d > d'$ means that the priority of operation d is higher than the priority of operation d'.

In analogy with the previous case, define $B_{alg}(k)$ as the set of permutations sorted by the partial left-greedy algorithm with k decreasing stacks (from now on it will be called the almost left-greedy $D^k I$ machine). We notice immediately that the permutation 231, which is not sorted by the left-greedy $D^k I$ machine, is instead sortable by the almost left-greedy DI machine: the sequence of the operations performed by the algorithm in this case is $d_0, d_0, d_1, d_1, d_0, d_1, d_2, d_2, d_2$. Unfortunately, in this case $B_{alg}(k)$ is not in general a permutation class, except for the case $k = 1$, for which it is quite easy to prove that the almost left-greedy strategy is equivalent to the (optimal) sorting strategy defined in [5], so that $B_{alg}(1) = Av(3241, 3142)$. As an example, for $k = 2$, the permutation 631425 can be sorted, whereas its subpermutation 52314 cannot.

The fact that $B_{alg}(k)$ is not a downset in general makes the analysis of the almost left-greedy machine more difficult. However, when $k = 2$, we are able to obtain a partial characterization of $B_{alg}(2)$ in terms of *barred patterns*.

Theorem 3. *1. Let π be an almost left-greedy $D^2 I$ sortable permutation; then:*

- *π avoids* 3214*;*
- *π avoids the following barred patterns, each of which is obtained by suitably adding barred elements to the pattern* 52314*:*
  - $63\bar{1}425$*;*
  - $7\bar{2}\bar{1}4536$, $7\bar{3}\bar{1}4526$*;*
  - $\bar{7}\bar{2}8\bar{1}4536$, $\bar{7}\bar{3}8\bar{1}4526$*;*
  - $\bar{8}\bar{2}7\bar{1}4536$, $\bar{8}\bar{3}7\bar{1}4526$*.*

2. Let π be a permutation that is not almost left-greedy $D^2 I$ sortable. Then one of the following cases holds:

- *π contains* 3214*;*
- *π contains one of the barred patterns listed above;*
- *π contains an occurrence of* 52314 *that extends to* 82714536 *(resp.,* 83714526*) which in turn is part of one of the following patterns:*
  - 9 3 1 8 2 5 6 4 7 *(resp.,* 9 4 1 8 2 5 6 3 7*);*
  - 10 2 1 4 9 3 6 7 5 8 *(resp.,* 10 2 1 5 9 3 6 7 4 8*);*

– 10 3 1 4 9 2 6 7 5 8 *(resp.,* 10 3 1 5 9 2 6 7 4 8*);*
– 10 2 11 1 4 9 3 6 7 5 8 *(resp.,* 10 2 11 1 5 9 3 6 7 4 8*);*
– 10 3 11 1 4 9 2 6 7 5 8 *(resp.,* 10 3 11 1 5 9 2 6 7 4 8*);*
– 11 2 10 1 4 9 3 6 7 5 8 *(resp.,* 10 2 11 1 5 9 3 6 7 4 8*);*
– 11 3 10 1 4 9 2 6 7 5 8 *(resp.,* 10 3 11 1 5 9 2 6 7 4 8*);*

The above theorem fails to characterize $B_{alg}(2)$, due the long "bad" permutations listed in (2). Unfortunately the construction used to obtain these "bad" patterns can be repeated to generate, starting from 52314, a sequence of permutations of increasing lengths whose sortability depends on how many times we iterate the construction. To be more precise, define the permutations $\gamma_m \in S_{3m+2}$ as follows:

$$\gamma_m = 3m+2,\ \underbrace{2\ 3m+1\ 1}_{231},\ \underbrace{4\ 3m\ 3}_{231} \ldots \underbrace{2m-2\ 2m+3 2n-3}_{231}\ \underbrace{2m\ 2m+1\ 2m-1}_{231} 2m+2.$$

In other words, starting from $\gamma_1 = 52314$, γ_{i+1} is obtained from γ_i by inserting a new maximum in the first position and putting the old maximum between 2 and 1, then suitably rescaling the remaining elements. We can prove that:

1. $\gamma_i \leq \gamma_{i+1}$, for each $i \geq 1$;
2. $\gamma_i \in B_{alg}(2)$ if and only if i is even.

For example, we have:

- $\gamma_1 = 52314 \notin B_{alg}(2)$;
- $\gamma_2 = 82714536 \in B_{alg}(2)$;
- $\gamma_3 = 11\ 2\ 10\ 14936758 \notin B_{alg}(2)$;

  $\vdots$

The existence of an infinite sequence of permutations with this property suggests that it would be quite difficult to obtain simple characterization of $B_{alg}(2)$; it is also conceivable that it should be possible to adapt the above construction to greater values of k, thus obtaining similar (negative) results.

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A Family of Bell Transformations

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This talk is based on joint work with Daniel Birmajer and Michael D. Weiner

Aiming at developing a unifying approach for a variety of enumeration problems, and in the spirit of the work by E. T. Bell on partition polynomials, we introduce a family of sequence transformations defined via partial Bell polynomials.

Let a, b, c, d be fixed. Given a sequence $x = (x_n)_{n\in\mathbb{N}}$, we let $y = \mathscr{Y}_{a,b,c,d}(x)$ be the sequence defined by

$$y_n = \sum_{k=1}^{n} \frac{1}{n!}\left[\prod_{j=1}^{k-1}(an+bk+cj+d)\right] B_{n,k}(1!x_1, 2!x_2, \dots) \text{ for } n \ge 1, \tag{1}$$

where $B_{n,k}$ denotes the (n,k)-th (exponential) partial Bell polynomial. We call $\mathscr{Y}_{a,b,c,d}(x)$ the Bell transform of x with parameters (a,b,c,d).

For $k = 0,1,2,\dots$, the polynomials $B_{n,k}(z_1, z_2, \dots, z_{n-k+1})$ may be defined through the series expansion

$$\frac{1}{k!}\left(\sum_{j=1}^{\infty} z_j \frac{t^j}{j!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}(z_1, z_2, \dots)\,\frac{t^n}{n!}.$$

These polynomials are homogeneous of degree k, of weight n, and they can be written as

$$B_{n,k}(z_1,\dots,z_{n-k+1}) = \sum_{\alpha\in\pi(n,k)} \frac{n!}{\alpha_1!\alpha_2!\cdots}\left(\frac{z_1}{1!}\right)^{\alpha_1}\left(\frac{z_2}{2!}\right)^{\alpha_2}\cdots,$$

where $\pi(n,k)$ denotes the set of multi-indices $\alpha \in \mathbb{N}_0^{n-k+1}$ such that

$$\alpha_1 + \alpha_2 + \cdots = k \text{ and } \alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots = n.$$

Note that $B_{n,k}$ contains as many monomials as the number of partitions of $[n] = \{1,\dots,n\}$ into k parts. Thus, if x enumerates some class of building blocks (with x_j distinct blocks of type j), then the sequence $\mathscr{Y}_{a,b,c,d}(x)$ counts the number of objects that can be made from these building blocks by placing them (according to their type) on a set of partitions induced by the parameters (a,b,c,d). Moreover, the term

$$\frac{1}{n!}\left[\prod_{j=1}^{k-1}(an+bk+cj+d)\right] B_{n,k}(1!x_1, 2!x_2, \dots)$$

gives the number of such objects made with exactly k blocks. For example, if the induced partitions consist of interval blocks, then the set of resulting objects of length n made with k such blocks is given by

$$\tfrac{k!}{n!} B_{n,k}(1!x_1, 2!x_2, \dots). \tag{2}$$

This corresponds to $(a,b,c,d) = (0,1,-1,1)$. The sum over $k = 1,\dots,n$ then gives the INVERT transform of x, see e.g. [1, 3, 6], and the quantity (2) may be interpreted as the number of colored compositions of n with k parts, where part j comes in x_j different colors.

Another special case is the NONCROSSING PARTITION transform, introduced by Beissinger in [1] and systematically studied by Callan in [5]. It corresponds to $(a,b,c,d) = (1,0,-1,1)$, giving $\prod_{j=1}^{k-1}(an+bk+cj+d) = \frac{n!}{(n-k+1)!}$. In this case, (1) becomes

$$y_n = \sum_{k=1}^{n} \tfrac{1}{(n-k+1)!} B_{n,k}(1!x_1, 2!x_2, \dots),$$

which counts the configurations obtained by placing the building blocks enumerated by x on top of the noncrossing partitions of $[n]$. In particular, if $x = \mathbb{1} = (1,1,\dots)$, then

$$y_n = \sum_{k=1}^{n} \tfrac{1}{(n-k+1)!} B_{n,k}(1!, 2!, \dots) = \sum_{k=1}^{n} \tfrac{1}{n}\tbinom{n}{n-k}\tbinom{n}{k-1} = \tfrac{1}{n+1}\tbinom{2n}{n}.$$

Thus $\mathscr{Y}_{1,0,-1,1}(\mathbb{1})$ is the sequence of Catalan numbers that enumerates noncrossing partitions Dyck paths, rooted trees, and many others combinatorial objects.

The family $\mathscr{Y}_{a,b,c,d}$ also includes several of the known transformations studied by Bernstein and Sloane in [3]. For example, EXP, REVERT, and CONV are instances of BELL and can therefore be treated with the unifying approach provided by the partial Bell polynomials.

We study $\mathscr{Y}_{a,b,c,d}$ from the algebraic and combinatorial points of view. We give formulas for the inverse $\mathscr{Y}_{a,b,c,d}^{-1}$ and provide equivalent forms of (1) in terms of generating functions. These results are obtained using Lagrange inversion together with certain interpolating properties of the partial Bell polynomials proved in [4].

We will end with some examples (focusing on rooted planar maps and certain classes of permutations) that illustrate how $\mathscr{Y}_{a,b,c,d}$ may be used to link the enumeration of certain classes of combinatorial structures with the enumeration of building blocks that serve as "primitive elements" within each class.

Main theorems

Theorem 1 (Inverse relations). *Let $x = (x_n)_{n\in\mathbb{N}}$, $y = \mathscr{Y}_{a,b,c,d}(x)$, $q_{n,k}(t) = t\prod_{j=1}^{k-1}(an + dj + t)$.*

(i) If $c \neq 0$, then

$$x_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{n!}\left[\frac{q_{n,k}(b+c) - q_{n,k}(b)}{c}\right] B_{n,k}(1!y_1, 2!y_2, \dots).$$

(ii) *If $c = 0$, then*

$$x_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{n!} q'_{n,k}(b) B_{n,k}(1!y_1, 2!y_2, \dots),$$

where $q'_{n,k}$ denotes the derivative $\frac{d}{dt} q_{n,k}$.

Theorem 2 (Generating functions). *Let $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ be sequences such that $y = \mathscr{Y}_{a,b,c,d}(x)$. Let $X(t) = \sum_{n=1}^{\infty} x_n t^n$ and $Y(t) = \sum_{n=1}^{\infty} y_n t^n$.*

(i) *If $c \neq 0$ and $d \neq 0$,*

$$X\left(t\big(1 + dY(t)\big)^{a/d}\right) = \frac{1}{c}\left[1 - \big(1 + dY(t)\big)^{-c/d}\right]\big(1 + dY(t)\big)^{-b/d}.$$

(ii) *If $c = 0$ and $d \neq 0$,*

$$X\left(t\big(1 + dY(t)\big)^{a/d}\right) = \log\left(\big(1 + dY(t)\big)^{1/d}\right)\big(1 + dY(t)\big)^{-b/d}.$$

(iii) *If $c \neq 0$ and $d = 0$,*

$$X\left(te^{aY(t)}\right) = \frac{1}{c}\left[1 - e^{-cY(t)}\right]e^{-bY(t)}.$$

(iv) *If $c = d = 0$,*

$$X\left(te^{aY(t)}\right) = Y(t)e^{-bY(t)}.$$

Selected combinatorial applications

Rooted planar maps

In [11] Tutte studied the enumeration of rooted planar maps and established a link to the enumeration of nonseparable rooted planar maps, which under the action of $\mathscr{Y}_{2,0,-1,1}$ may be considered their prime elements.

Let a_n be the number of rooted planar maps with n edges and let b_n be the number of nonseparable rooted planar maps with n edges. Let $A(t)$ and $B(t)$ be the generating functions for $a = (a_n)_{n\in\mathbb{N}}$ [10, A000168] and $b = (b_n)_{n\in\mathbb{N}}$ [10, A000139], respectively. As proved by Tutte (cf. [11, Equation 6.3]), these functions satisfy the functional equation

$$A(t) = B\big(t(1 + A(t))^2\big),$$

which implies $a = \mathscr{Y}_{2,0,-1,1}(b)$.

Bicubic maps

Another example amenable to BELL transformations is given by the following connection between rooted bicubic planar maps and their subclass of 3-connected elements. In [11] Tutte observed that "Each rooted bicubic map can be represented as a multiple extension of a 3-connected bicubic core" and proved the functional equation

$$F(t) = G\big(t(1+F(t))^3\big), \tag{3}$$

where $F(t) = \sum f_n t^n$ enumerates the rooted bicubic maps of $2n$ vertices (cf. [10, A000257]) and $G(t) = \sum g_n t^n$ counts those maps that are 3-connected. This means $f = \mathscr{Y}_{3,0,-1,1}(g)$. In [11] Tutte proved that $f = (f_n)_{n\in\mathbb{N}}$ is given by

$$f_n = \frac{3(2n-1)!2^n}{(n-1)!(n+2)!} \text{ for } n \geq 1. \tag{4}$$

This sequence also gives the number of indecomposable σ-avoiding permutations of length n for any $\sigma \in \{1342, 2413, 2431, 3142, 3241, 4132, 4213\}$.

Indecomposable permutations

Let S_n be the set of permutations on $[n]$, and let p denote the sequence of factorials $(n!)_{n\in\mathbb{N}}$. The inverse of p under the INVERT transform $\mathscr{Y}^{-1}_{0,1,-1,1}(p)$ is the sequence A003319 in [10] that enumerates the class of indecomposable permutations on $[n]$. This reflects the fact that every permutation in S_n can be split into indecomposable permutations of length less than or equal to n, so they play the role of building blocks from which all permutations can be constructed. In fact, every permutation on $[n]$ may be represented as a composition of n whose parts of length j are labeled by indecomposable permutations of length j.

Using this interpretation it is easy to see that, if a pattern σ is indecomposable, then every permutation in $\mathsf{Av}(\sigma)$ can be split into σ-avoiding indecomposable permutations (denoted by $\mathsf{Av}^{\mathsf{ind}}(\sigma)$). Thus, if $a_\sigma = (\mathsf{Av}_n(\sigma))_{n\in\mathbb{N}}$ and $i_\sigma = (\mathsf{Av}^{\mathsf{ind}}_n(\sigma))_{n\in\mathbb{N}}$, then

$$i_\sigma = \mathscr{Y}^{-1}_{0,1,-1,1}(a_\sigma) \quad \text{(cf. [7, Lem. 3.1]).}$$

For example, if $\sigma \in \{2413, 2431, 3142, 3241, 4132, 4213\}$, then $\mathsf{Av}^{\mathsf{ind}}_n(\sigma) = f_n$ like in (4), and

$$\mathsf{Av}_n(\sigma) = \sum_{k=1}^{n} \frac{k!}{n!} B_{n,k}(1!f_1, 2!f_2, \dots).$$

Also, since A001519 enumerates $\mathsf{Av}(\sigma,\tau)$ for $(\sigma,\tau) \in \{(321, 2341), (321, 3412), (321, 3142)\}$, we can use $\mathscr{Y}^{-1}_{0,1,-1,1}$ to obtain $\mathsf{Av}^{\mathsf{ind}}_n(\sigma,\tau) = 2^{n-1}$.

Corresponding formulas may be obtained for classes avoiding two patterns of length 4. For example, if (σ,τ) is any of the following pairs of permutations:

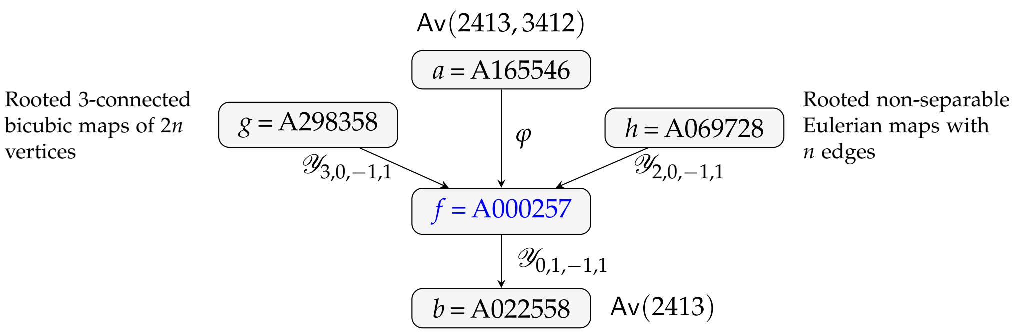
(4321, 4312), (4312, 4231), (4312, 4213), (4312, 3412), (4231, 4213),
(4213, 4132), (4213, 4123), (4213, 2413), (3142, 2413),

it is known [8] that $\mathsf{Av}_n(\sigma,\tau)$ is the large Schröder number A006318($n-1$). As a consequence, we have that $\mathsf{Av}_n^{\mathsf{ind}}(\sigma,\tau)$ is given by the little Schröder number A001003($n-1$).

The class $\mathsf{Av}(\mathbf{2413},\mathbf{3412})$

Bell transformations can be combined with the OEIS [10] to create flows of sequences associated with a particular sequence of interest. This often leads to combinatorial connections that can be verified through the functional equation satisfied by the generating functions.

Let us illustrate this strategy with the sequence A000257 (number of rooted bicubic maps of $2n$ vertices, number of rooted Eulerian maps with n edges, or $\mathsf{Av}_n^{\mathsf{ind}}(2413)$, for instance). Using Bell transformations one can discover the following connections:



where $\varphi = \mathscr{Y}_{1,1,-1,1} \circ R$ and $R \circ (x_1, x_2, \dots) = (1, x_1, x_2, \dots)$. The identities $f = \mathscr{Y}_{3,0,-1,1}(g)$, $f = \mathscr{Y}_{2,0,-1,1}(h)$, $b = \mathscr{Y}_{0,1,-1,1}(f)$ are consistent with the building block approach, and therefore expected. However, the connection between A000257 and A165546 is surprising.

First, since $\mathscr{Y}^{-1}_{1,1,-1,1} = \mathscr{Y}_{-1,0,-1,-1}$, the conjectured identity $f = \varphi(a)$ is equivalent to

$$R(a) = \mathscr{Y}_{-1,0,-1,-1}(f). \tag{5}$$

If $A(t)$ is the generating function for the sequence a, then the right-shifted sequence $R(a)$ has generating function $t(1 + A(t))$. Hence (5) together with Corollary 2 give the functional equation

$$1 + F\big(t(1 - t(1 + A(t)))\big) = \frac{1}{1 - t(1 + A(t))}.$$

Now, using $F(t) = \frac{1}{32t^2}\big(-1 + 12t - 24t^2 + (1-8t)^{3/2}\big)$ one can verify the identity

$$16t^2(1 + F(t))^2 - (8t^2 + 12t - 1)(1 + F(t)) + t^2 + 11t - 1 = 0,$$

which leads to the functional equation

$$t^4(1 + A(t))^3 + (5t^3 - 11t^2)(1 + A(t))^2 + (3t^2 + 10t - 1)(1 + A(t)) - 9t + 1 = 0.$$

The validity of this equation was recently proved by Miner and Pantone [9]. In other words, we do have $f = \big(\mathscr{Y}_{1,1,-1,1} \circ R\big)(a)$ and as a consequence, we get the formulas

$$f_n = \sum_{k=1}^{n} \binom{n+k}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!, 2!a_1, 3!a_2, \dots), \text{ and}$$
$$a_{n-1} = \sum_{k=1}^{n} \binom{-n-2}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!f_1, 2!f_2, \dots) \text{ for } n \geq 2.$$

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Permutations Sorted by a Finite and an Infinite Stack in Series

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This talk is based on joint work with Murray Elder

The research interest in pattern avoiding permutations is inspired by Donald Knuth's work in stack-sorting. According to Knuth, a permutation can be sorted by passing through a single infinite stack if and only if it avoids a sub-permutation pattern 231 [3]. Murphy extended Knuth's research by using two infinite stacks in series and found out that the basis for generated permutations is infinite [4] but Elder proved that the basis is finite when one of the stack is limited to depth two and the number of basis is 20 permutations [1]. An example of sorting permutation using two stack in series is in Figure 1

We prove that the set of permutations sorted by a stack of depth $t \geq 3$ and an infinite stack in series has infinite basis, by constructing an infinite antichain of unsortable permutations [2]. This answers an open question on identifying the point at which, in a sorting process with two stacks in series, the basis changes from finite to infinite.

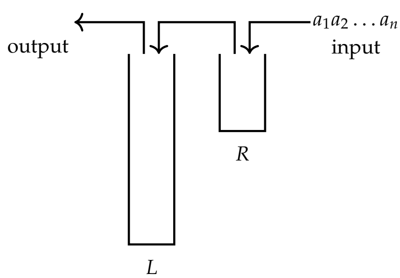

Figure 1: A stack R of depth t and an infinite stack L in series

A simple lemma then implies the result for depth 4 or more. A computer search by the authors has yielded 8194 basis permutations of lengths up to 13 (see Table 1); basis permutations are listed at

`https://github.com/gohyoongkuan/stackSorting-3`.

The antichain used to prove our theorem was found by examining this data and looking for patterns that could be arbitrarily extended.

As a by product, we also can find an explicit antichain for in the basis of $S(t, \infty)$.

Table 1: Number of basis elements for $S(3,\infty)$ of length up to 13

Permutation length	Number of sortable permutations	Number of basis elements
5	120	0
6	711	9
7	4700	83
8	33039	169
9	239800	345
10	1769019	638
11	13160748	1069
12	98371244	1980
13	737463276	3901

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In [1], Ira Gessel and Yan Zhuang have coined the concept of *shuffle-compatibility*: a property shared by many (but not all) known and less-known permutation statistics. In this abstract, which is an outline of the paper-in-progress [3], we shall apply this concept to the *exterior peak set* statistic, proving a conjecture of Gessel and Zhuang, and furthermore study variants of this concept.

Definitions and the main theorem

We let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. For each $n \in \mathbb{Z}$, we set $[n] = \{1, 2, \ldots, n\}$.

If $n \in \mathbb{N}$, then an *n-permutation* shall mean an n-tuple of distinct positive integers. For example, $(3, 6, 4)$ and $(9, 1, 2)$ are 3-permutations, but $(2, 1, 2)$ is not.

A *permutation* means an n-permutation for some $n \in \mathbb{N}$. This concept of permutation (inherited from [1]) is nonstandard; however, our later definition of permutation statistics will ensure that the extra liberty to use arbitrary positive integers as entries does not significantly impact the results.

If π is an n-permutation for some $n \in \mathbb{N}$, then we refer to n as the *size* of π and denote it by $|\pi|$. Furthermore, we say that π is *nonempty* if $n > 0$, and we use the notation π_i for the i-th entry of π.

If $n \in \mathbb{N}$, then two n-permutations α and β are said to be *order-equivalent* if every $i, j \in [n]$ satisfy the logical equivalence $(\alpha(i) < \alpha(j)) \iff (\beta(i) < \beta(j))$.

A *permutation statistic* is a map st from the set of all permutations to an arbitrary set that has the following property: Whenever α and β are two order-equivalent permutations, we have $\operatorname{st}(\alpha) = \operatorname{st}(\beta)$. Thus, a permutation statistic can alternatively be viewed as a statistic defined on the set of all permutations in the usual sense (i.e., all permutations of the sets $[n]$ for $n \in \mathbb{N}$), because each permutation (in the sense above) is order-equivalent to a unique permutation of its size (in the usual sense).

Examples of permutation statistics are Des (sending each permutation π to its set $\operatorname{Des} \pi$ of descents), maj (sending each permutation to its major index) and inv (sending each permutation to its number of inversions). A more elaborate example is the *peak set* statistic Pk; it sends each n-permutation π to the set $\operatorname{Pk} \pi$ of all *peaks* of π, which are the elements $i \in \{2, 3, \ldots, n-1\}$ satisfying $\pi_{i-1} < \pi_i > \pi_{i+1}$. This statistic has been studied by Aguiar, Nyman, Petersen and others. Two variants are the *left peak statistic* Lpk (which is defined just as Pk, but i now ranges over $\{1, 2, \ldots, n-1\}$ instead of $\{2, 3, \ldots, n-1\}$, and π_0 is understood to be 0) and the *right peak statistic* Rpk (defined similarly). We refer to [1] or [3] for precise definitions.

The permutation statistic that we shall mainly focus on is the *exterior peak set* Epk. It sends each n-permutation π to the set $\operatorname{Epk}\pi$ of all *exterior peaks* of π, which are the elements $i \in [n]$ satisfying $\pi_{i-1} < \pi_i > \pi_{i+1}$, where both π_0 and π_{n+1} are understood to be 0. Equivalently, $\operatorname{Epk}\pi$ is the number of appearances of the consecutive patterns 132 and 231 in the word $0\pi0$ (that is, π flanked by zeroes on both sides). For example,

$$\operatorname{Epk}(1,4,3,2,9,8) = \{2,5\}; \qquad \operatorname{Epk}(3,1,4,2) = \{1,3\};$$
$$\operatorname{Epk}(1,2,3,4) = \{4\}.$$

Two permutations π and σ are said to be *disjoint* if no number appears in both π and σ.

If π and σ are disjoint permutations with sizes $m = |\pi|$ and $n = |\sigma|$, then a *shuffle* of π and σ means an $(m+n)$-permutation in which both π and σ appear as subsequences. For instance, the shuffles of the two disjoint permutations $(3,1)$ and $(2,6)$ are

$$(3,1,2,6), \qquad (3,2,1,6), \qquad (3,2,6,1),$$
$$(2,3,1,6), \qquad (2,3,6,1), \qquad (2,6,3,1).$$

This naive definition of a shuffle does not attempt to deal with equal numbers, but suffices for what we shall do in the following.

A permutation statistic st is said to be *shuffle-compatible* if and only if it has the following property: For any two disjoint permutations π and σ, the multiset

$$\{\operatorname{st}(\tau) \mid \tau \text{ is a shuffle of } \pi \text{ and } \sigma\}$$

depends only on $\operatorname{st}(\pi)$, $\operatorname{st}(\sigma)$, $|\pi|$ and $|\sigma|$.

Our main theorem – conjectured by Gessel and Zhuang in [1] – is:

Theorem 1. *The permutation statistic* Epk *is shuffle-compatible.*

This joins the ranks of a series of similar theorems about other statistics proven in [1]. In particular, [1] showed that the statistics Des (descent set), des (number of descents), maj (major index), Pk (peak set), Lpk (left peak set), Rpk (right peak set) and several others are shuffle-compatible, whereas the statistics inv (number of inversions), $\operatorname{des} + \operatorname{maj}$ (the sum of the number of descents and the major index) and various others are not. Shuffle-compatibility is not equivalent to defining a subalgebra of descent algebras (for example, Epk does not define such a subalgebra).

The proof of Theorem 1 relies on a generalization of Stanley's concept of P-partitions and its closest relatives (Stembridge's enriched P-partitions and Petersen's left enriched P-partitions). We refer to [3] for details.

Left and right shuffles

We can refine the concept of shuffles. Namely, if π and σ are two disjoint nonempty permutations, then a shuffle of π and σ is called a *left shuffle* (of π and σ) if it begins with π_1; otherwise it is a *right shuffle*. We can now define two finer versions of shuffle-compatibility:

- A permutation statistic st is said to be *left-shuffle-compatible* if for any two disjoint nonempty permutations π and σ having the property that

$$\text{the first entry of } \pi \text{ is greater than the first entry of } \sigma, \tag{1}$$

  the multiset $\{\operatorname{st}(\tau) \mid \tau \text{ is a left shuffle of } \pi \text{ and } \sigma\}$ depends only on $\operatorname{st}(\pi)$, $\operatorname{st}(\sigma)$, $|\pi|$ and $|\sigma|$.

- A permutation statistic st is said to be *right-shuffle-compatible* if for any two disjoint nonempty permutations π and σ having the property (1), the multiset $\{\operatorname{st}(\tau) \mid \tau \text{ is a right shuffle of } \pi \text{ and } \sigma\}$ depends only on $\operatorname{st}(\pi)$, $\operatorname{st}(\sigma)$, $|\pi|$ and $|\sigma|$.

We now claim:

Theorem 2. *The permutation statistic* Epk *is left-shuffle-compatible and right-shuffle-compatible.*

Our proof of Theorem 2 (again, for details see [3]) involves a detour through QSym, the ring of quasisymmetric functions. It uses four additional binary operations on QSym, introduced in [2].

We also prove that the statistics Des (descent set), des (descent number) and Lpk (left peak set) are left-shuffle-compatible and right-shuffle-compatible, but the statistics Rpk (right peak set) and maj (major index) are not.

Descent statistics and the QSym connection

The concept of shuffle-compatibility is closely related to the $\mathbb{Q}$-algebra QSym of quasisymmetric functions, as Gessel and Zhuang already observed in [1]. Let us outline the connection. (We refer to [5, Section 7.19] or [4, Chapter 5] for the definition of QSym.)

If $n \in \mathbb{N}$, then each subset $I = \{i_1 < i_2 < \cdots < i_k\}$ of $[n-1]$ determines a composition of n (namely, the composition $(i_1 - i_0, i_2 - i_1, \ldots, i_{k+1} - i_k)$, where we set $i_0 = 0$ and $i_{k+1} = n$). This latter composition is denoted by $\operatorname{Comp} I$. This defines a bijection Comp from the set of all subsets of $[n-1]$ to the set of all compositions of n. When this map is applied to the descent set $\operatorname{Des}\pi$ of an n-permutation π, we denote the resulting composition $\operatorname{Comp}(\operatorname{Des}\pi)$ by $\operatorname{Comp}\pi$.

A permutation statistic st is said to be a *descent statistic* if and only if $\operatorname{st}\pi$ (for π a permutation) depends only on $\operatorname{Comp}\pi$. In other words, st is a descent statistic if and only if every two permutations π and σ satisfying $\operatorname{Comp}\pi = \operatorname{Comp}\sigma$ satisfy $\operatorname{st}\pi = \operatorname{st}\sigma$.

All shuffle-compatible permutation statistics currently known (including Des, des, maj, Lpk, Rpk and Epk) are descent statistics. For example, Epk is a descent statistic, since every positive integer n and every n-permutation π satisfy $\operatorname{Epk}\pi = \{i \in \operatorname{Des}\pi \cup \{n\} \mid i -$
(and both $\operatorname{Des}\pi$ and n can be recovered from $\operatorname{Comp}\pi$).

For any descent statistic st, we define the *kernel* $\mathcal{K}_{\mathrm{st}}$ of st to be the $\mathbb{Q}$-vector subspace of QSym spanned by all elements of the form $F_{\operatorname{Comp}\pi} - F_{\operatorname{Comp}\sigma}$, where π and σ are two permutations of the same size satisfying $\operatorname{st}\pi = \operatorname{st}\sigma$, and where F stands for Gessel's fundamental basis of QSym (so F_α is what is denoted by L_α in [5, Proposition 7.19.1] or [4, Definition 5.2.4]). Then, a descent statistic st is shuffle-compatible if and only if its kernel $\mathcal{K}_{\mathrm{st}}$ is an ideal of QSym. (This is explicitly stated in [3], but the idea goes back to [1].)

Thus, shuffle-compatible descent statistics correspond to a certain kind of ideals of QSym. The quotients of QSym by these ideals are called *shuffle algebras* in [1].

In [3], we find two spanning sets for the kernel $\mathcal{K}_{\mathrm{Epk}}$ of Epk. One is in terms of the fundamental basis; the other in terms of the monomial basis. Similar descriptions can probably be found for kernels of other prominent descent statistics.

The quasisymmetric point of view also illuminates left-shuffle-compatibility: We show that a descent statistic st is left-shuffle-compatible and right-shuffle-compatible if and only if its kernel $\mathcal{K}_{\mathrm{st}}$ is an ideal of the *dendriform* algebra QSym, which relies on the dendriform operations introduced in [2]. This fact, along with certain identities for these dendriform operations, is key to the proof of Theorem 2.

While the above results (particularly Theorem 1) can be viewed as a conclusion of [1], several questions arise from the work, and much remains to be done.

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The Prolific Proportion of Permutations

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This talk is based on joint work with Simon R. Blackburn, Peter Winkler

The *breadth* of an n-permutation π is the minimum value of the pairwise manhattan distance between entries. In this talk we present several related results on permutation breadth and connections to large pattern containment. In particular we find the probability that a large random permutation has a given breadth, and calculate the expected breadth of a random permutation.

Background

Definition 1. The *breadth* of a permutation π, denoted $\mathsf{br}(\pi)$, is defined as

$$\mathsf{br}(\pi) := \min_{i \neq j}\{|i-j| + |\pi(i) - \pi(j)|\}.$$

Let π be a permutation of length n. For $1 \leq k \leq n$, let $\Delta_k(\pi)$ be the set of patterns of length $n-k$. That is,

$$\Delta_k(\pi) = \{\sigma \in \mathcal{S}_{n-k} : \sigma \prec \pi\}.$$

It follows that $|\Delta_k(\pi)| \leq \binom{n}{k}$, and we say that a permutation π is *k-prolific* if we have $|\Delta_k(\pi)| = \binom{n}{k}$. That is, π is k-prolific if all the possible k-deletions result in a unique pattern. For example, the permutation $123\cdots n$ is not 1-prolific, since deleting any entry produces the same pattern. If $\pi = 3142$, then π is 1-prolific since $\Delta_1(\pi) = \{132, 213, 231, 312\}$, but not 2-prolific since $\Delta_2(\pi) = \{12, 21\}$. See Figure 1 for examples of minimal 5- and 6-prolific permutations.

Breadth is connected to prolificity through the following theorem:

Theorem 2 ([1])**.** *A permutation is k-prolific if and only if it has breadth of at least $k+2$. Additionally, k-prolific n-permutations exist for every $n \geq \lceil k^2/2 \rceil + 2k + 1$.*

For a permutation π, we say that the pair i, j is a *k-close pair* if $|i-j| + |\pi(i) - \pi(j)| = k$. Previously, Wolfowitz studied the distribution of 2-close pairs within a random permutation π: entries that are adjacent both in position and value. In particular, he proved the following theorem:

Theorem 3 (Wolfowitz [3])**.** *For a large random permutation π, the number of 2-close pairs is asymptotically Poisson distributed with mean 2. It follows that the probability that a random permutation has no 2-close pairs is e^{-2}.*

Corollary 4. *The probability that a random permutation has breadth strictly greater than 2 (and hence is 1-prolific) is e^{-2}.*

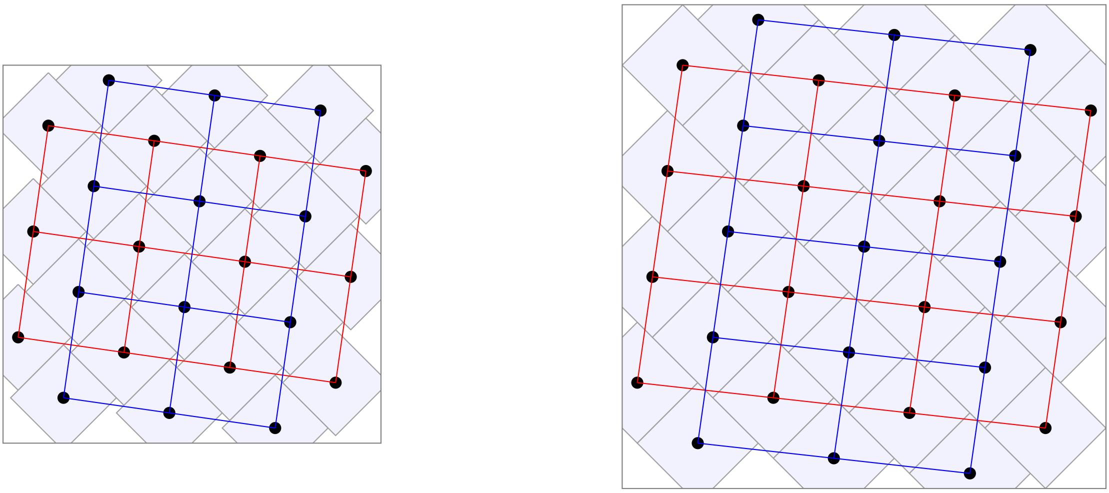

Figure 1: The plots of a minimal 5-prolific permutation of length 24 and a minimal 6-prolific permutation of length 31, represented as a pairs of interlocking grids of entries.

Agenda

In this talk, we present several probabilistic results related to the breadth of random permutations. Specifically, for fixed k, we examine the limiting probability that a (uniformly) random permutation is k-prolific by approximating the distribution of $(k-1)$-close pairs of entries. Then, we calculate the expected breadth of a large random permutation using probabilistic techniques and the Bonferroni inequalities.

Specifically, we present the following theorems:

Theorem 5 ([2])**.** *The number X of $(k+1)$-close pairs in a large random permutation is Poisson-distributed with mean k^2+k. That is,*

$$\mathbb{P}\left[X=m\right]=e^{-k^2-k}\frac{(k^2+k)^m}{m!}.$$

In particular, the probability that a permutation has breadth at least $k+2$ (i.e., is k-prolific) is e^{-k^2-k}.

Theorem 6 ([2])**.** *The expected breadth of a large random permutation π is*

$$\mathbb{E}\left[\mathsf{br}(\pi)\right]=1+\sum_{k=0}^{\infty}e^{-k^2-k}\approx 2.13782018.$$

We prove Theorem 5 by investigating the distribution of $(k-1)$-close pairs in a random permutation. Using a moment method, we show that this distribution is asymptotically Poisson, which allows us to estimate the number of permutations with breadth at least k.

However, knowing the distribution of k-close pairs for fixed k is insufficient to prove Theorem 6. For that, we need a series of carefully constructed bounds to show convergence, and we measure a closely related statistic: the *minimum jump* (denoted mj) of a permutation:

$$\mathsf{mj}(\pi) = \min_i \{|\pi(i+1) - \pi(i)|\},$$

and calculate the expected minimum jump of a random permutation to be

$$\mathbb{E}\left[\mathsf{mj}(\pi)\right] = \sum_{k=0}^{n} e^{-2k} \approx 1.15651764.$$

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Enumerative and Algebraic Combinatorics of OEIS A071356

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In [4, 6], the study of pattern avoiding inversion sequences was initiated. In [7], Martinez and Savage ask if $\mathbf{I}_n(e_i > e_j \leq e_k)$ is counted by OEIS `A071356`. We provide a generating function argument that this is indeed the case. The sequence OEIS `A071356` is known to count certain underdiagonal lattice paths [1]. We find an explicit bijection between $\mathbf{I}_n^R(e_i \geq e_j < e_k)$ and the underdiagonal lattice paths.

Rephrasing the results of [3] provides a surjective map τ from S_n to all Dyck paths. Moreover, the fibers of τ are intervals in the weak order and coincide with a lattice congruence on S_n. When τ is restricted to the top elements of the interval, we get a bijection between $S_n(\underline{13}2)$ and Dyck paths. In [5], we show that $S_n(2\underline{14}3, 3\underline{14}2, \underline{14}23, \underline{14}32)$ is counted by OEIS `A071356`. Additionally, we find a surjective map ρ from S_n to the underdiagonal lattice paths whose fibers are intervals in the weak order and coincide with a lattice congruence on S_n. When ρ is restricted to the top elements of the interval, we get a bijection between $S_n(2\underline{14}3, 3\underline{14}2, \underline{14}23, \underline{14}32)$ and the underdiagonal lattice paths.

In the next section, we provide a different surjective map ω from S_n to the underdiagonal lattice paths. We prove that the fibers of ω are intervals in the weak order. While the fibers of ω do not coincide with a lattice congruence on the weak order, they do refine the fibers of τ. In a later section, we show that the Lehmer codes of the top elements of the intervals are precisely the elements of $\mathbf{I}_n^R(e_i \geq e_j < e_k)$. So ω restricted to the top elements is the desired bijection.

The map $\omega : S_n \to \mathcal{RSP}_n$

Let S_n be the symmetric group consisting of permutations of $[n]$. Throughout this section, we assume $\sigma \in S_n$ is written in one-line notation as $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$.

Definition 1. We call a pair (σ_i, σ_j) an *inversion* of σ if $i < j$ and $\sigma_i > \sigma_j$.

Definition 2. A *non-Dyck inversion* of a permutation $w \in S_n$ is an inversion (σ_i, σ_j) such that there exists some σ_k where $i < j < k$ and $\sigma_j < \sigma_k < \sigma_i$. A *Dyck inversion* of a permutation $\sigma \in S_n$ is an inversion (σ_i, σ_j) that is not a non-Dyck inversion.

Definition 3. Let $\mathcal{D}_n$ be the set of Dyck paths, that is, underdiagonal paths in a $n \times n$ box that use north and east steps of length 1. Let $\mathcal{SP}_n$ be the set of Schröder paths, that is underdiagonal paths in a $n \times n$ box consisting of north (length 1), east (length 1), and diagonal northeast (length $\sqrt{2}$) steps. Let $\mathcal{RSP}_n \subseteq \mathcal{SP}_n$ be the set of *restricted Schröder paths*, where there are no diagonal steps on the main diagonal and every diagonal step is immediately followed by an east step.

Definition 4. Let the *height* of an east step of a Dyck path $p \in \mathcal{D}_n$ be defined as its distance from the southern boundary of the $n \times n$ box.

Definition 5. Let $\tau : S_n \to \mathcal{D}_n$ be the following map. Suppose $\sigma \in S_n$. We begin with the $n \times n$ square. The square can be considered a grid with n columns. If we consider the n-tuple $(d_1, \cdots, d_n)$, where

$$d_i = |\{j \mid (\sigma_i, \sigma_j) \text{ is a Dyck inversion}\}|$$

we place a $d_i \times 1$ rectangle into the i^{th} column so that the upper left corner aligns with the point $(i-1, i-1)$. Note that by construction, all boxes lie below the diagonal. Equivalently, $\tau(\sigma)$ can be considered the unique Dyck path where the east step in the ith column occurs at height $i - d_i + 1$. The fact that the resulting lower boundary of the boxes is a Dyck path is proven in [5].

Remark 6. The map τ is inspired by [2]. In fact, when τ is restricted to the bottom elements we recover the main bijection of [2].

Definition 7. Fix $\sigma \in S_n$ and suppose $\sigma_i = \sigma_j - 1$ and $\sigma_k = \sigma_j + 1$. If $1 < j < n$, we define σ_j to be *NEE-positive* if $i < j$ and $k < j$. If $1 < j < n$, we define σ_j to be *atomic* if $i > j$ and $k > j$. Additionally, $\sigma_j = 1$ is *atomic* if $k > j$ and $\sigma_j = n$ is *atomic* if $i > j$.

Definition 8. Each Dyck path has a natural *edge labeling* as follows. We label the unique east edge in the ith column with i. For each east edge there is a corresponding unique north edge. It is found by traveling in the northeast direction from the center of the east edge until hitting the path again. The point at which we hit the path again is the center of the corresponding north edge. We label the north edge with the same label as its corresponding east edge.

Definition 9. Fix $p \in \mathcal{D}_n$, and let j be the label of a fixed east step. If $1 < j < n$, we define j to be *NEE-positive* if the east step labeled with $j+1$ has the same height as the east step labeled with j and the east step labeled with $j-1$ has a smaller height. If $1 < j < n$, we define j to be *atomic* if the east step labeled with $j-1$ has the same height as the east step labeled with j and the east step labeled with $j+1$ has a larger height. Additionally, 1 is *atomic* if the east step labeled with 2 has nonzero height and n is *atomic* if $n-1$ has the same height as n. We will often refer to atomic edge labels as *atoms*.

Proposition 10. *Fix $\sigma \in S_n$. If σ_j is NEE-positive or atomic, then the east step labeled with σ_j in $\tau(\sigma)$ will be NEE-positive or atomic respectively. Fix $p \in \mathcal{D}_n$. If the east step labeled with i in p is NEE-positive or atomic, then for all σ such that $\tau(\sigma) = p$, i will be NEE-positive or atomic respectively.*

Proposition 11. *Fix $\sigma \in S_n$. Let b_i and b_{i+1} be atomic edge labels in $\tau(\sigma)$ such that $b_i < b_{i+1}$ and there is no atomic b_j with $b_i < b_j < b_{i+1}$. Then there exists exactly one label a_{i+1} in between b_i and b_{i+1} that is NEE-positive.*

Corollary 12. *Let $\sigma \in S_n$ be fixed. If σ has k NEE-positive instances, then it has $k+1$ atoms. Moreover, if we arrange the NEE-positive and atoms in increasing order as a tuple, they will alternate. Let this tuple be $(b_0, a_1, b_1, a_2 \ldots, b_{k-1}, a_k, b_k)$.*

Definition 13. For fixed $\sigma \in S_n$. Let $\{b_0, b_1, \cdots, b_k\}$ be the atoms of $\tau(\sigma)$ listed in increasing order. We define the *partner atom function* $\gamma_\sigma : \{b_0, b_1, \ldots, b_{k-1}\} \to \{b_1, \ldots, b_k\}$ as follows. If $\gamma_\sigma(b_i) = b_j$, we refer to b_j as the *partner atom* of b_i. If σ is clear from the context, we will omit the σ subscript. Let $\text{left}(a, b)$ be whichever of a and b appears first in σ when interpreting the one-line notation of σ as a linear extension on $[n]$. γ_σ is defined recursively as follows. Our base case is $\gamma(b_{k-1}) = b_k$. For $0 \le m < k-1$, $\gamma(b_m) = \text{left}(b_{m+1}, \gamma(b_{m+1}))$

Definition 14. To each permutation $\sigma \in S_n$ with k NEE-positive instances, we can associate a certain $0-1$ vector of length k. We define the map $\beta : S_n \to \{0,1\}^k$ as follows. For $0 \le j \le k-1$, the $(j+1)^{th}$ entry of $\beta(\sigma)$ is 1 if $\text{left}(b_j, \gamma(b_j)) = \gamma(b_j)$. The $(j+1)^{th}$ entry of $\beta(\sigma)$ is 0 if $\text{left}(b_j, \gamma(b_j)) = b_j$.

Definition 15. Let $\omega : S_n \to \mathcal{RSP}_n$ be the following map. Let $\sigma \in S_n$. Start with the Dyck path $\tau(\sigma)$. If the ith entry of $\beta(\sigma)$ is nonzero, add a triangle (translation of the convex hull of $(0,0), (0,1), (1,1)$) in the a_i^{th} column so that the edge $(0,1)$-$(1,1)$ agrees with the bottom edge of the lowest box in the a_i^{th} column (where a_i is the label of the well-defined NEE-positive label as in Corollary 12). The resulting lower boundary of the shape will be a restricted Schröder path with the desired properties.

Definition 16. Fix $\sigma \in S_n$. Suppose that $\sigma_i\underline{\sigma_j\sigma_k}\sigma_m$ is an instance of a $1\underline{24}3$ pattern. We call such a pattern *left inversion dense* or *LID* if

$$\left|\{\sigma_u \mid i < u, \sigma_i > \sigma_u\}\right| > \left|\{\sigma_u \mid j < u, \sigma_j > \sigma_u\}\right|.$$

Definition 17. Fix $\sigma \in S_n$. Suppose that $\sigma_i\underline{\sigma_j\sigma_k}\sigma_m$ is an instance of a LID $1\underline{24}3$ pattern. We say that such a pattern is *partner atom neutral* or *PAN* if either

- at least one of σ_j and σ_k is not atomic
- both σ_j and σ_k are atomic and $\gamma(\sigma_j) \ne \sigma_k$

Proposition 18. *Let $x \lessdot y$ in the weak order. Then $\omega(x) = \omega(y)$ if and only if y is obtained from x by one of the following moves.*

- *A $2\underline{14}3$ to $2\underline{41}3$ move*
- *A $3\underline{14}2$ to $3\underline{41}2$ move*
- *A $4\underline{13}2$ to $4\underline{31}2$ move*
- *A $1\underline{24}3$ to $1\underline{42}3$ move where $1\underline{24}3$ is PAN in x and $1\underline{42}3$ is PAN in y*

Theorem 19. *The fibers of ω are intervals.*

Corollary 20. *Fix a fiber of ω. The top element is a permutation that avoids $2\underline{14}3, 3\underline{14}2, 4\underline{13}2$ patterns as well as PAN $1\underline{24}3$ patterns. The bottom element is a permutation that avoids $2\underline{41}3, 3\underline{41}2, 4\underline{31}2$ patterns as well as PAN $1\underline{42}3$ patterns.*

Proposition 21. *ω is surjective.*

Definition 22. Let $MAv_n = S_n(4\underline{13}2, 3\underline{14}2, 2\underline{14}3, 1\underline{24}3_{LID})$ be the set of permutations avoiding the normal vincular patterns $4\underline{13}2, 3\underline{14}2, 2\underline{14}3$ and LID $1\underline{24}3$ patterns.

Definition 23. Let $Av_n = S_n(4\underline{13}2, 3\underline{14}2, 2\underline{14}3, 1\underline{24}3_{PAN})$ be the set of permutations avoiding the normal vincular patterns $4\underline{13}2, 3\underline{14}2, 2\underline{14}3$ and PAN $1\underline{24}3$ patterns.

Lemma 24. *$MAv_n \subseteq Av_n$.*

Theorem 25. $|S_n(2\underline{14}3, 3\underline{14}2, \underline{14}23, \underline{14}32)| = |\mathcal{RSP}_n|$

Theorem 26. $|S_n(2\underline{14}3, 3\underline{14}2, \underline{14}23, \underline{14}32)| = |Av_n|$

Corollary 27. $|Av_n| = |\mathcal{RSP}_n|$

Corollary 28. *ω restricts to a bijection from Av_n to $\mathcal{RSP}_n$.*

Avoiders in Inversion Sequences

We consider the family of avoiders on inversion sequences known as class 1064 in [7].

Definition 29. We recall from [7] that elements of $\mathbf{I}_n(e_i > e_j \leq e_k)$ take the following form:

$$e_1 \leq \cdots \leq e_t > e_{t+1} > \cdots > e_n$$

for some t such that $1 < t \leq n$. Let t be called the *peak* of such an inversion sequence.

In [7], it is noted that true unimodal inversion sequences (class 1265 in [7]) are in bijection with $S_n(2143, 3142, 4132)$ via Lehmer codes. In this spirit, we will find it useful to consider $\mathbf{I}_n^R(e_i \geq e_j < e_k)$, the reversed inversion sequences of class 1064. Let L^{-1} be the inverse of the map that takes permutations to their Lehmer codes.

Proposition 30. $L^{-1}(\mathbf{I}_n^R(e_i \geq e_j < e_k)) \subseteq MAv_n$

Proof. Fix $\mathbf{e} = (e_1, \cdots, e_n) \in \mathbf{I}_n^R(e_i \geq e_j < e_k)$ and let $\sigma \in S_n$ be the unique permutation such that the Lehmer code of σ is $\mathbf{e}$. Assume for the sake of contradiction that σ has one of the forbidden four subpatterns of MAv_n. Suppose the forbidden subpattern is $\sigma_i - \sigma_j\sigma_k - \sigma_m$. We note that $e_i \geq e_j$ for the normal vincular patterns (follows from $\sigma_i > \sigma_j$) and by definition in the LID pattern. Since σ_j and σ_k form an ascent, we find that $e_j \leq e_k$ and the existence of σ_m shows that $e_j < e_k$. Thus, we would have an instance of the pattern $e_i \geq e_j < e_k$, which contradicts $\mathbf{e} \in \mathbf{I}_n^R(e_i \geq e_j < e_k)$. This gives the desired result. □

Definition 31. Let $I_{n,t} \subseteq \mathbf{I_n}(e_i > e_j \leq e_k)$ be the avoiding inversion sequences of length n with peak t. Let $C_s = \{(e_1 \leq \cdots \leq e_s) \in \mathbb{N}^s \mid e_i \leq i-1\}$. Let $A_{s,k} = \{(e_{s+1} > e_{s+2} > \cdots > e_{s+k}) \in \mathbb{N}^k \mid e_{s+1} \leq s\}$.

It is shown in [7], that $|C_s| = c_s$, the sth Catalan number.

Theorem 32. *The generating function of* $\mathbf{I}_n^R(e_i \geq e_j < e_k)$ *is*

$$\frac{2x+1-\sqrt{1-4x-4x^2}}{4x}.$$

Proof. There exists a surjective two-to-one map $\alpha : \bigcup_{s=1}^{\infty} \bigcup_{k=0}^{s+1} C_s \times A_{s,k} \to I_{s+k}$ for $s \geq 1$. The generating function for twice the codomain is equal to the generating function for the domain. After adjusting the initial terms properly, this yields:

$$\sum_{n=0}^{\infty} 2a_n x^n = 2 + x + \sum_{s=1}^{\infty} c_s x^s \left[\binom{s+1}{0} x^0 + \binom{s+1}{1} x^1 + \cdots + \binom{s+1}{s+1} x^{s+1} \right]$$

$$2\sum_{n=0}^{\infty} a_n x^n = 2 + x + \sum_{s=1}^{\infty} c_s x^s (1+x)^{s+1} = 2 + x + (1+x) \sum_{s=1}^{\infty} c_s [x(1+x)]^s$$

We recall that the Catalan numbers have generating function $\sum_{t=0}^{\infty} c_t y^t = \frac{1-\sqrt{1-4y}}{2y}$. Using the Catalan generating function for $y = x(1+x)$, and after a routine computation, we find the desired generating function:

$$\sum_{t=0}^{\infty} a_t x^t = \frac{2x+1-\sqrt{1-4x(x+1)}}{4x}$$

□

Theorem 33. *The generating function of* $\mathcal{RSP}_n$ *is*

$$\frac{2x+1-\sqrt{1-4x-4x^2}}{4x}.$$

Corollary 34. $Av_n = L^{-1}(\mathbf{I}_n^R(e_i \geq e_j < e_k))$

Proof. We note that $Av_n \supseteq L^{-1}(\mathbf{I}_n^R(e_i \geq e_j < e_k))$ by Proposition 30 and Lemma 24. By Theorem 32, Corollary 27, and Theorem 33, the two sets have the same generating function hence the same sizes so are equal. □

Corollary 35. *The map* ω *restricts to a bijection between* $L^{-1}(\mathbf{I}_n^R(e_i \geq e_j < e_k))$ *and* $\mathcal{RSP}_n$.

Proof. This follows from Theorem 19, Corollary 20, and Corollary 34. □

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Connecting Descent and Peak Polynomials

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Introduction

Denote by $\mathfrak{S}_n$ the symmetric group of permutations $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ of $[n] = \{1, 2, \ldots, n\}$ written in one-line notation. We will draw the graph of σ by plotting points (i, σ_i) and connecting consecutive points.

We define the descent set of σ as follows:

$$\mathrm{Des}(\sigma) = \{i \mid \sigma_i > \sigma_{i+1}\} \subset [n-1].$$

Note that the descents mark the beginnings of the intervals where the graph is decreasing, as seen in Figure 1 below.

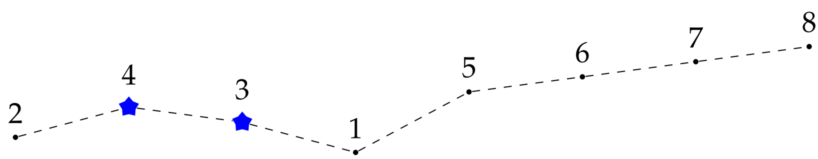

Figure 1: The graph of $\sigma = 34215678$ with descents marked in blue.

For a given set S and $n > \max(S)$, we let $D(S, n)$ be the set of all permutations in $\mathfrak{S}_n$ with descent set S, and put $d(S, n) = |D(S, n)|$. In 1915, it was shown by MacMahon [5] that this is a polynomial in n. More recently, Diaz-Lopez et al. [3] proved this polynomial expands into the binomial basis around $n - m$, where $m = \max(S)$, and gave a combinatorial interpretation for the coefficients. Using the notation $\sigma|_i = \{\sigma_1, \sigma_2, \ldots, \sigma_i\}$, we give a version of their result slightly altered to include the case $m > \max(S)$ as follows:

Theorem 1 ([3]). For any finite set of positive integers S with $\max(S) \leq m$ we have:

$$d(S, n) = a_0(S)\binom{n-m}{0} + a_1(S)\binom{n-m}{1} + \cdots + a_m(S)\binom{n-m}{m}, \qquad (1)$$

where the constant $a_k(S)$ is the number of $\sigma \in D(S, 2m)$ such that:

$$\sigma|_m \cap [m+1, 2m] = [m+1, m+k].$$

The underlying idea is quite elegant and will be useful to show a similar result for peak polynomials. Simply put, if $\sigma \in D(S, n)$ has $\{\sigma_1, \sigma_2, \ldots, \sigma_m\} \cap [m+1, n] = A$ for some k element set A, and B is any other k element subset of $[m+1, n]$, then exchanging elements of A and B while preserving the orders gives another permutation with descent set S. Therefore, for each k, it is enough to count for the simplest k-element subset of $[m+1,n]$ and multiply with $\binom{n-m}{k}$.

For example, there are 3 elements in $D(\{2,3\},n)$ satisfying $\sigma|_m \cap [5,8] = \varnothing : 14325678, 2431567$
So we have $a_0(\{2,3\}) = 3$ for $m = 4$. Calculating the other coefficients similarly, we obtain:

$$d(\{2,3\},n) = 3\binom{n-4}{0} + 8\binom{n-4}{1} + 7\binom{n-4}{2} + 2\binom{n-4}{3} + 0\binom{n-4}{4}. \quad (2)$$

Another well studied permutation statistic is given by peak point. Here we define the peak points and their counterpart valley points of the partition to be the points higher and lower than their neighbors respectively:

$$\text{Peak}(\sigma) = \{\sigma \mid \sigma_i > \sigma_{i+1}, \sigma_{i-1}\} \subset [n-1]/\{1\},$$

$$\text{Valley}(\sigma) = \{\sigma \mid \sigma_i < \sigma_{i+1}, \sigma_{i-1}\} \subset [n-1]/\{1\}.$$

The example $\sigma = 34215678$ from Figure 1 has $\text{Peak}(\sigma) = \{2\}$ and $\text{Valley}(\sigma) = \{4\}$. We also set $\text{Spike}(\sigma) = \text{Peak}(\sigma) \cup \text{Valley}(\sigma)$ to be the set of all extremal points that are not corner points.

For a given set I and $n > \max(I)$, we let $P(I,n)$ be the set of permutations in $\mathfrak{S}_n$ with peak set I, and set $p(I,n) = 2^{-n+|I|+1}|P(I,n)|$. Note that peaks are more restrictive in the sense that $p(I,n) = 0$ if I contains 1 or any consecutive entries. For the rest of this work, we will focus our attention to *admissible* peak sets I: $I \subset [n-1]/\{1\}$ such that $i \in I \Rightarrow i+1 \notin I$.

In [1] Billey, Burzdy and Sagan proved that $p(I,n)$ is a polynomial in n, and conjectured that the coefficients of this polynomial in a binomial basis centered at $\max(I)$ are non-negative. Their conjecture was proved in 2017 by Diaz-Lopez et al. [2] using the recursion of peaks, without describing the actual coefficients.

In this work, we tie the theory of peak and descent polynomials together by giving a binary expansion of $d(S,n)$ in terms of peak polynomials. We use this expansion to give a description of the peak polynomial coefficients analogous to the one in Theorem 1. In the next section, we extend our notions of descents and peaks to $\mathfrak{B}_n$, the set of marked permutations of n with $2^n n!$ elements. The added exponent of 2 cancels out with the $2^{-n+|I|+1}$ from the peak polynomial definition, giving us a way to expand descent polynomials in terms of peak polynomials. In a later section, we define involutions on permutations that flip the descents on an initial section and we use them to partition permutations with a given descent set to calculate the coefficients for the peak polynomial.

Descents and Peaks of Marked Permutations

We start our section by tweaking our notation a little bit to express our formulas easier. Note that the peaks and valleys of a permutation only depend on its descent set. In fact for any $S \in [n-1]$, we can talk about the peaks and valleys of S:

$$\text{Peak}(S) = \{1 < i \leq n-1 \mid i \in S, i-1 \notin S\},$$

$$\text{Valley}(S) = \{1 < i \leq n-1 \mid i \notin S, i-1 \in S\}.$$

Note that with this notation, $\text{Peak}(\sigma) = \text{Peak}(\text{Des}(\sigma))$ and $\text{Valley}(\sigma) = \text{Valley}(\text{Des}(\sigma))$ as expected. We also set $\text{Spike}(S) := \text{Peak}(S) \cup \text{Valley}(S)$.

Denote by $\mathfrak{B}_n$ the set of signed permutations:

$$\mathfrak{B}_n := \{\rho = \rho_1\rho_2\ldots\rho_n \mid \forall i \leq n \exists k : \rho_k = i \text{ or } \rho_k = -i\}.$$

Note that the definitions of descent, peak, spike and valley naturally extend to signed permutations by saying i is a descent of ρ if $\rho_i > \rho_{i+1}$.

Lemma 2 ([4]). Let $\sigma \in S_n$ have $\text{Peak}(\sigma) = I$. Denote by $\text{Mark}(\sigma)$ the 2^n element subset of $\mathfrak{B}_n$ that give σ when marks are erased. Then, for all $\rho \in \text{Mark}(\sigma)$, $\text{Spike}(\rho) \supset I$. Conversely, for any $S \subset [n-1]$ satisfying $\text{Spike}(S) \supset I$, there are exactly $2^{|I|+1}$ elements in $\text{Mark}(\sigma)$ with descent set equal to S.

Theorem 3. We have $d(S,n) = \sum_{I \subset \text{Spike}(S)} p(I,n)$.

For example:

- $d(\varnothing, n) = p(\varnothing, n) = 1$.
- $d(\{1\}, n) = p(\{2\}, n) + p(\varnothing, n)$.
- For $1 < k < n$, $d(\{k\}, n) = p(\{k\}, n) + p(\{k+1\}, n) + p(\varnothing, n)$.
- $d(\{k, k+1, \ldots, k+j\}, n) = p(\{k, k+j+1\}, n) + p(\{k\}, n) + p(\{k+j+1\}, n) + p(\varnothing, n)$.

For any set $I \in [n]/\{1\}$, we will let S_I denote the unique subset of $[n]/\{n\}$ satisfying $\text{Spike}(S_I) = I$, constructed by alternating the elements of I to be peaks and valleys such that the rightmost one is not a peak. For example, for $I = \{2,4\}$ we have $S_I = \{2,3\}$: the descent set with a peak at 2 and a valley at 4.

Corollary 4. For any admissible set I, $p(I,n) = \sum_{J \subset I} (-1)^{|I|-|J|} d(S_J, n)$.

If we consider our running example $I = \{2,4\}$, we get the following formulas from Theorem 3 and Corollary 4 respectively:

$$d(\{2,3\},n) = p(\{2,4\},n) + p(\{2\},n) + p(\{4\},n) + p(\varnothing,n), \quad (3)$$

$$p(\{2,4\},n) = d(\{2,3\},n) - d(\{1\},n) - d(\{1,2,3\},n) + d(\varnothing,n). \quad (4)$$

A combinatorial expression for peak coefficients

We start with defining an operation on permutations that 'flips' the orders of some initial coordinates.

Definition 5. Let $\sigma \in S_n$. Let $i \leq n$, and $\sigma|_i = \{a_1 < a_2 < \cdots < a_i\}$. We define the involution fl_i as follows:

$$\mathrm{fl}_i(\sigma)_j = \begin{cases} a_{i-k+1} & j \leq i, \sigma_j = a_k \\ \sigma_j & j > i. \end{cases}$$

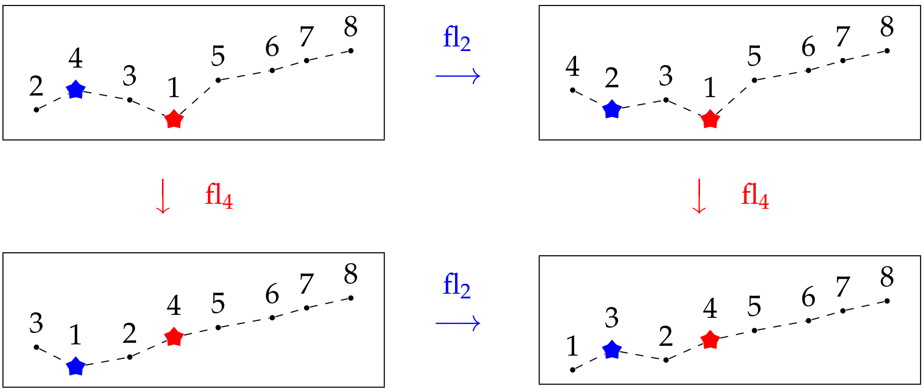

Figure 2: 2 and 4 flips of $\sigma = 21478536$ admits a 6-flip.

Remark 6. The involution fl_i satisfies the following:

- $\mathrm{fl}_i(\sigma)|_i = \sigma|_i$.
- For $k < i$, k is a descent of $\mathrm{fl}_i(\sigma)$ iff it is not a descent of σ.
- For $k > i$, k is a descent of $\mathrm{fl}_i(\sigma)$ iff it is a descent of σ.
- fl_i exchanges all the peaks less than $i-1$ with valleys, and all the valleys less than $i-1$ with peaks.

In Figure 2, we see an instance of operations fl_2 and fl_4 commuting. This in fact is true in general.

Proposition 7. For all i, j, fl_i and fl_j commute.

For any admissible set $I = \{i_1, i_2, \ldots, i_k\}$, we put $\mathrm{fl}_I := \mathrm{fl}_{i_1} \circ \mathrm{fl}_{i_2} \circ \cdots \circ \mathrm{fl}_{i_k}$. This operation is well-defined by the proposition above.

We say σ admits an i-flip if $i \in \mathrm{Spike}(\sigma)$ and $\mathrm{Spike}(\mathrm{fl}_i(\sigma)) = \mathrm{Spike}(\sigma)/\{i\}$. Visually, this means that fl_i straightens out the peak or valley point at i. For example, 21478536 from Figure 2 admits a 4-flip, but not a 2-flip.

Corollary 8. For all i, j such that $|i-j| > 1$, σ admits an i-flip if and only if $\mathrm{fl}_j(\sigma)$ admits an i-flip.

	fl_2	fl_4
14325678	✗	✓
24315678	✗	✓
34215678	✓	✓

$k = 0$

	fl_2	fl_4
15324678	✗	✓
15423678	✗	✗
25314678	✗	✗
25413678	✗	✗
35214678	✓	✗
35412678	✗	✗
45213678	✓	✗
45312678	✓	✗

$k = 1$

	fl_2	fl_4
16523478	✗	✗
26513478	✗	✗
36512478	✗	✗
46512378	✗	✗
56213478	✓	✗
56312478	✓	✗
56412378	✓	✗

$k = 2$

	fl_2	fl_4
57612348	✗	✗
675123478	✓	✗

$k = 3$

Table 1: The elements $\sigma \in D(\{2,3\},8)$ satisfying $\sigma|_4 \cap [5,8] = [5,4+k]$.

Lemma 9. For any admissible set I and any $J \subset I$, the operation fl_J gives a bijection between elements of $D(S_{I/J}, n)$ and elements of $D(S_I, n)$ that admit a j-flip for all $j \in J$. In particular, for any $m \geq \max(I)$ we have:

$$\sigma|_m \cap [m+1, 2m] = [m+1, m+k] \iff \mathrm{fl}_J(\sigma)|_m \cap [m+1, 2m] = [m+1, m+k].$$

Theorem 10. For any admissible set of I with $\max(I) \leq m$ we have

$$P(S,n) = b_0(S)\binom{n-m}{0} + b_1(S)\binom{n-m}{1} + \cdots + b_m(S)\binom{n-m}{m}, \tag{5}$$

where the constant $b_k(S)$ is the number of $\sigma \in D(S, 2m)$ such that:

$$\sigma|_m \cap [m+1, 2m] = [m+1, m+k],$$

and σ does not admit any i-flips.

We will end this section by calculating the expansion of $p(\{2,4\}, n)$. Recall that $S_{\{2,4\}} = \{2,3\}$. For all elements $\sigma \in D(\{2,3\},8)$ satisfying $\sigma|_4 \cap [5,8] = [5,4+k]$ for some k we need to check if σ admits a 2-flip or a 4-flip. Checking for 2-flips is very straightforward, we just need to check whether $\sigma_1 > \sigma_3$. 4-flips are slightly more tricky as fl_4 does not simply exchange a pair of coordinates, and we actually need to calculate $\mathrm{fl}_4(\sigma)$ to see if $\mathrm{fl}_4(\sigma)_3$ is smaller than σ_5. For each k, the related permutations σ can be found in Table 1, along with the information on whether they admit 2 or 4-flips.

Counting the elements that admit neither 2 nor 4-flips from Table 1 gives us the following formula:

$$p(\{2,4\},n) = 0\binom{n-4}{0} + 4\binom{n-4}{1} + 4\binom{n-4}{2} + 1\binom{n-4}{3}.$$

In fact, the inclusion-exclusion principle allows us to read the coefficients for $p(2,n)$(ones that admit only 4-flips), $p(4,n)$(ones that admit only 2-flips) and $p(\varnothing,n)$(ones that admit only 4-flips) from Table 1:

$$\begin{aligned} p(\{2\},n) &= 2\binom{n-4}{0} + 1\binom{n-4}{1} + 0\binom{n-4}{2} + 0\binom{n-4}{3}, \\ p(\{4\},n) &= 0\binom{n-4}{0} + 3\binom{n-4}{1} + 3\binom{n-4}{2} + 1\binom{n-4}{3}, \\ p(\varnothing,n) &= 1\binom{n-4}{0} + 0\binom{n-4}{1} + 0\binom{n-4}{2} + 0\binom{n-4}{3}. \end{aligned}$$

Note that $p(\{2,4\},n) + p(\{2\},n) + p(\{4\},n) + p(\varnothing,n) = d(\{2,3\},n)$ as required.

Acknowledgements

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Permutation Packing in Words of the Form $\pi\pi^r$ and $\pi\pi$

Julia Krull[1], *Eric Redmon*[2], *Andrew Reimer-Berg*[3]

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In 2014, Cratty, Erickson, and Negassi studied pattern avoidance in words of the form $\pi\pi$ [3]. In 2015, Anderson, Diepenbroek, and Stoll explored pattern avoidance in words of the form $\pi\pi^r$ [1]. Both of these groups discovered several orderly ways to count pattern-avoiding words due to the symmetries of the words they considered.

Instead of avoiding patterns, we study pattern packing; that is, we identify words with as many copies of a pattern as possible. While Burstein, Hästö and Mansour [2] studied packing patterns into general words, our focus is to pack patterns into words with symmetries. In particular, given a pattern ρ, we consider how many times we can pack ρ into words of the form $\pi\pi^r$ and $\pi\pi$, what the ρ-optimal words of these forms look like, and how many ρ-optimal words exist for a given length of π.

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The Principal Möbius Function of Permutations With Opposing Adjacencies

David Marchant The Open University

This talk is based on joint work with Robert Brignall

We show that if a permutation π has two opposing adjacencies, then the value of the principal Möbius function $\mu[1, \pi]$ is zero. Further, we show that asymptotically the percentage of permutations with opposing adjacencies is bounded below by 39%.

To the best of our knowledge, this is the first time that a positive proportion of the permutation poset has been shown to have a specific principal Möbius function value.

The set of all permutations is a poset under the partial order given by containment. A closed interval $[\sigma, \pi]$ in a poset is the set $\tau : \sigma \leq \tau < \pi$, and a half-open interval $[\sigma, \pi)$ is the set $\tau : \sigma \leq \tau < \pi$. The Möbius function is defined recursively on an interval of a poset $[\sigma, \pi]$ as:

$$\mu[\sigma, \pi] = \begin{cases} 0 & \text{If } \sigma \not\leq \pi \\ 1 & \text{If } \sigma = \pi \\ -\sum_{\lambda \in [\sigma, \pi)} \mu[\sigma, \lambda] & \text{otherwise} \end{cases}$$

We define an *adjacency* in a permutation to be a pair of adjacent points of the permutation that have the form $(i, i+1)$ or $(i, i-1)$. As examples, 367249815 has two adjacencies, 67 and 98; and 1432 also has two adjacencies, 43 and 32. If an adjacency is ascending, then it is an *up-adjacency*, otherwise it is a *down-adjacency*.

If a permutation π contains at least one up-adjacency, and at least one down-adjacency, then we say that π has *opposing adjacencies*.

We prove the following:

Theorem 1. *If π has opposing adjacencies, then $\mu[1, \pi] = 0$.*

Kaplansky [2] gives an expression for the probability that a permutation of length n has exactly r adjacencies. Corteel, Louchard and Pemantle [1] show that the distribution of adjacencies is Poisson. We combine the results from Kaplansky with those from Corteel, Louchard and Pemantle, and then use Theorem 1 to show that:

Theorem 2. *The percentage of permutations that have opposing adjacencies, and thus have principal Möbius function value equal to zero, is, asymptotically, bounded below by 39%.*

Following Theorem 2, it is natural to ask if we can find asymptotic bounds for the percentage of permutations that have principal Möbius function value equal to zero. Plainly, 39% is a lower bound. Based on numerical evidence supplied by Jason Smith [3], and calculations performed by the author, we conjecture that:

Conjecture 3. *The percentage of permutations that have principal Möbius function value equal to zero is bounded above by 61%.*

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Unknotted Cycles

Nathan McNew Towson University

This talk is based on joint work with Christopher Cornwell

When studying the structure of a permutation σ it is frequently useful to plot the permutation, treating the permutation as a function, placing a dot at $(i, \sigma(i))$ for each index i.

When one is interested in studying a permutation's cycle structure in addition to its structure as a sequence of numbers it is natural to extend the plot to be a *cycle diagram* [6]. For each index i we draw a vertical dotted line from the point (i, i) to the point $(i, \sigma(i))$, followed by a horizontal dotted line from $(i, \sigma(i))$ to $(\sigma(i), \sigma(i))$. If i is a fixed point, $i = \sigma(i)$, no additional lines are drawn. The result is a diagram in which the cycles of the permutation can be readily seen by tracing out the lines of the diagram in a natural way.

For example, from the cycle diagram of the permutation $\pi = 467513298$ (written in one line notation) depicted in Figure 1, one can readily identify the cycle decomposition $\pi = (145)(2637)(89)$ (written in cycle notation) by tracing out the lines of the diagram. Henceforth we always write permutations using one line notation. Note the only corners in a cycle diagram occur at the plotted points of the permutation and along the line $y = x$.

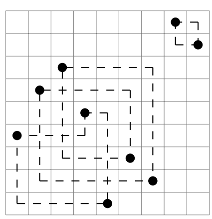


Figure 1: Cycle diagram of $\pi = 467513298$.

The appearance of these diagrams strongly resemble grid diagrams, which have recently become a useful tool in studying the structure of knots in topology. [3, 4, 8] Formally, a grid diagram is an $n \times n$ grid where every row and column has exactly two marked boxes and the entries in every row and column are connected by a dotted line. The diagram is then interpreted as a knot (an embedding of S_1 in $\mathbb{R}^3$) or a link (an embedding of multiple copies of S_1 if it has multiple components) by declaring all of the vertical lines to be overcrossings and the horizontal lines as undercrossings. (All of the vertical lines are interpreted as passing above the horizontal lines.)

It is known [3] that every knot has a grid diagram representation and there is a series of rules, "Cromwell moves" that can transform any grid diagram of a knot into any other grid diagram of the same knot.

Crucially, the only distinction between allowable cycle diagrams and allowable grid diagrams are:

1. In cycle diagrams one of the two designated points in each row/column must lie on the $y = x$ line, which isn't necessarily the case for grid diagrams.

2. Grid diagrams do not allow a single point in a row/column as occur as fixed points in cycle diagrams.

This means that the cycle diagram of any permutation without fixed points (a *derangement*) can be interpreted as the grid diagram of a corresponding link.

Definition 1. The **link associated to a cycle diagram** is the link obtained by drawing the cycle diagram of a derangement and then interpreting the diagram as the grid diagram of a link. In the case that the derangement is itself a cycle we will refer to this as the **knot associated to a cycle diagram**.

Note if a permutation is not a derangement (it has fixed points) then it is not associated to any link.

In the case that the knot associated to a cycle is the unknot (the knot type equivalent to a circle) we will refer to the cycle as an **unknotted cycle** for short (or respectively a permutation corresponding to an unlink as an **unlinked permutation**).

Example 2. Unknotted cycles can still have complicated-looking cycle diagrams. For example, the cycle 837295641 depicted below is unknotted, as the reader is invited to explore. This can be seen by removing the "kink" involving the indices 6 and 7 in the middle of the cycle, (recalling that vertical lines cross over horizontal lines) what is left is primarily one big kink that can be shrunk down until it too can be removed in a similar fashion. On the other hand the cycle 34512 is not an unknot, in fact it is a trefoil knot.

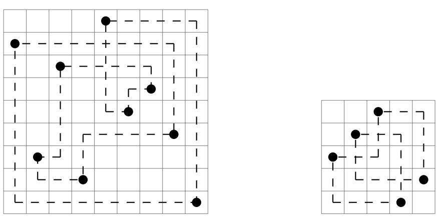


Figure 2: The cycle/grid diagrams for 837295641 (unknot) and 34512 (trefoil knot).

Main Results

Our main result is the following theorem about unknotted cycles. We denote by S_n the nth large Schröder number. The large Schröder numbers (sequence A006318 in the OEIS) are well-known to count a large number of combinatorial objects such as the separable permutations.

Theorem 3. *The number of the unknotted cycles of size $n+1$ is S_n, the nth Large Schröder number.*

The proof works by establishing a bijection to certain signed binary trees which are more easily counted. The crux of the argument is showing that the map from these trees to unknotted cycles is in fact surjective. This requires appealing to a result from the classification of Legendrian knots, known as Bennequin's innequality to exploit the structure of grid diagrams of the special form arising as cycle diagrams to establish the following, which is of interest in its own right.

Lemma 4. *If a cycle σ is an unknotted cycle, then there exists at least one index i for which $|\sigma(i) - i| = 1$. So the diagram of an unknotted cycle has at least one point immediately above or below the diagonal.*

By extending these ideas we are able to count all unlinked permutations (derangements). By an unlink, we mean the knot corresponding to each cycle of the permutation is an unknot in the grid diagram, and furthermore each of these unknots are not "linked" with one another. We obtain a bivariate generating function that keeps track of both the size of the permutation and the number of knots in the link.

Theorem 5. *Let $\mathcal{U}$ be the set of unlinked permutations (derangements), and denote by $c(\sigma)$ the number of cycles in σ (equivalently the knots in the link associated to σ). Define the bivariate generating function*

$$F(u,x) = 1 + \sum_{\sigma\in\mathcal{U}} u^{c(\sigma)} x^{|\sigma|}$$

to count unlinked permutations by length and number of components. Then $F(u,x)$ satisfies the recurrence

$$\frac{(2-ux)F(u,x) + ux^2F(u,x)^2 + uxF(u,x)\sqrt{1-6xF(u,x)+x^2F(u,x)^2}}{2} = 1$$

or equivalently $1 + (ux-2)F(u,x) + (1-ux-ux^2)F(u,x)^2 + (ux^2+u^2x^3)F(u,x)^3 = 0.$

Note the coefficient of u in this generating function is $\frac{1}{2}(x-x^2-x\sqrt{x^2-6x+1})$, the (shifted) generating function of the Large Schröder numbers. So this is in a sense a generalization of the Large Schröder numbers for unlinked permutations. Setting u=1, one recovers the generating function for the sequence of all unlinked permutations, a sequence, 1, 2, 8, 32, 143, 674, 3316, 16832... which does not yet appear in the OEIS.

Finally, we prove several results concerning the categorization of knots that can be associated to a cycle, and pose several open questions regarding the enumeration of other knots besides the unknot.

Signed Trees

Various authors [2, 9] introduce the idea of *separating trees* to study separable permutations. We introduce a similar structure, which seems to be the "right" way to keep track of the structure of an unknotted cycle.

Definition 6. A **rooted-signed-binary tree** is a rooted binary tree where each non-root node is given a sign (positive or negative). Furthermore we define two binary rooted trees to be equivalent if one can be obtained from another by a series of tree rotations, where tree rotations are allowed at a given node if either

1. The child node being rotated into the position of a parent node has the same sign as the parent.

2. The node is the root. In this case the new node is given the sign of the node rotated to the root.

For example these trees are all equivalent, and represent all the allowed rotations of the given tree.

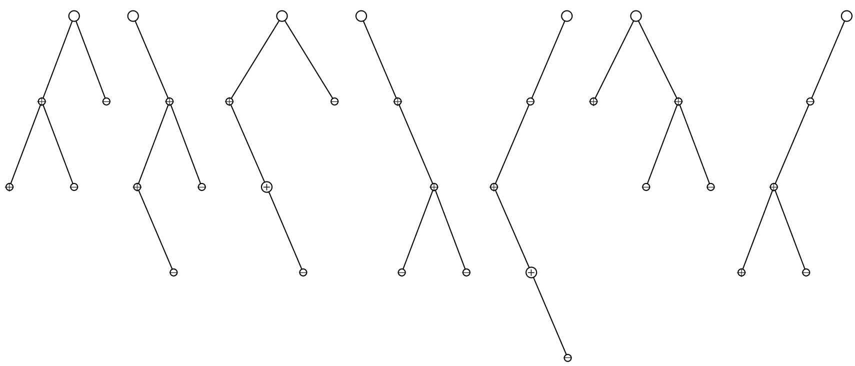


Figure 3: A complete set of equivalent rooted-signed-binary trees.

Proposition 7. *The rooted-signed-binary-trees are counted by the Large Schröder numbers.*

The bijection

We show the Large Schröder numbers count unknotted cycles by building a bijection between them and rooted-signed-binary trees. The idea of the bijection is as follows. Start with the rooted-signed-binary-tree containing only the unlabelled root, which corresponds to the trivial cycle, 21 which is clearly unknotted.

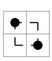


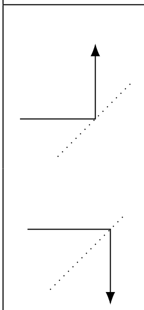
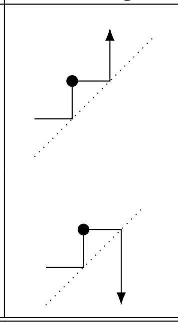
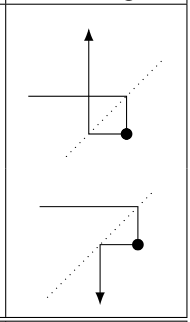
Figure 4: The cycle diagram of 21.

At any point in this construction a tree will have as many places where a new node can be added as the size of the cycle. To begin there are two places a node can be added, as either left or right children of the root node, corresponding to positions 1 and 2 of the cycle 21. When nodes are added, they affect the corresponding cycle according to these rules:

1. If a positive node is added in position i, insert the element $i+1$ prior to the element previously in position i. Increase all elements in the cycle that previously had value $i+1$ or greater by one.

2. If a negative node is added in position i, insert the element $i-1$ after the element previously in position i. Increase all elements in the cycle that previously had value $i-1$ or greater by one.

In terms of the cycle diagrams, this has the effect of taking one of the corners where the diagram made a right angle at the $y = x$ line and changing it into a notch or a kink. The change to the knot depends on the direction the lines in the diagram behaved prior to the insertion, summarized in the following table.

Before	Inserting $\oplus$	Inserting $\ominus$
		

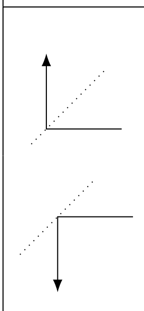
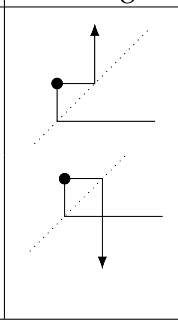
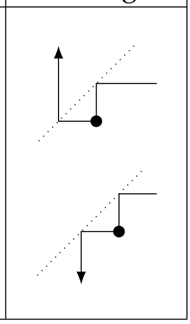
Before	Inserting $\oplus$	Inserting $\ominus$
		

Table 1: The insertion rules for signed nodes.

From these pictures we see that these changes will not affect the knot type of the cycle diagram into which they are inserted. Thus if a given cycle corresponds to an unknot before one of these operations is performed, it will still correspond to the unknot afterward as well.

As an example, we build up the cycle corresponding to the tree in Figure 3. We will construct the tree corresponding to the first diagram depicted, however the reader is invited to verify the same cycle is obtained for any of the diagrams irrespective of the order in which the nodes of the tree are considered.

Example 8. Consider the nodes from the first tree depicted in Figure 3 one at a time. Starting with the root, we have the trivial cycle 21, depicted in Figure 4. We first process the positive, right child of the root. Since the node is positive, and in position 1, we insert a 2 at the beginning, obtaining the cycle 213. We could also obtained this by looking at the cycle diagram, and noting that the corner in position 1 was a lower left corner (see Table 1) and replacing the corner in the diagram with the picture in the second column. At this point our cycle looks like the first cycle of Figure 5.

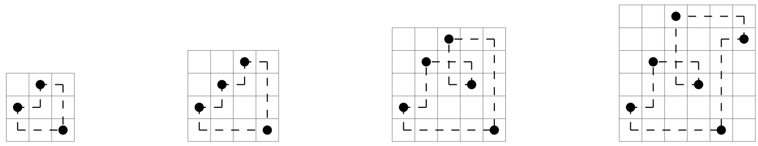


Figure 5: The cycles 231, 2341, 24531 and 246315

We now proceed by considering the leftmost leaf. It is positive and in relative position 1 of the tree, so we obtain the cycle 2341, shown second in figure 5. Since this node has no children, we move to the negative node right of that node. The two potential children of the previous node corresponded to positions 1 and 2 of the cycle, so this node occupies position 3. As it is negative, we insert a 3 after position 3 of the cycle, obtaining 24531, depicted third in figure 5. Last we consider the negative left child of the root. It is now in relative position 5, so we insert a 5 after the last position of the cycle, obtaining 246315.

Topology

To prove that the map from rooted-signed-binary trees to unknotted cycles is surjective we need some results from topology. The proof relies on Bennequin's inequality [1], or more precisely a reformulation of it for Legendrian knots [5] an important early result in modern contact geometry (see [7]). We begin with some notation.

Definition 9. Let σ be a derangement of n elements. Define (i, j) to be a *C-pair* if either $i < j < \sigma(i) < \sigma(j)$ or $i > j > \sigma(i) > \sigma(j)$. Additionally, define j to be a *UR-index* if $\sigma^{-1}(i) < i$ and $\sigma(i) < i$.

Remark 10. Note that (i, j) is a C-pair if and only if the union of the parts of the cycle diagram from node i to node $\sigma(i)$ and from j to $\sigma(j)$ has the form as on the left of Figure 6. Also, i is a UR-index if and only if the cycle diagram at node i appears as on the right of Figure 6.

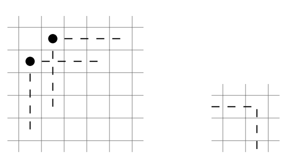


Figure 6: The cycle diagram at a C-pair (left) and UR-index (right).

Theorem 11 (Bennequin's Inequality)**.** *Let D_σ be the cycle diagram of σ and let K be the knot associated to D_σ (supposing therein that σ is a cycle). Define $C(D_\sigma)$ to be the number of C-pairs of D_σ and $UR(D_\sigma)$ the number of UR-indices. Let $g(K)$ be the Seifert genus of the knot. Then*

$$C(D_\sigma) - UR(D_\sigma) \leq 2g(K) - 1.$$

In our case, if K is an unknot, then it bounds a disk and so the Seifert genus is $g(K) = 0$, which allows us to relate the number of C-pairs to the number of UR-indices in an unknotted cycle.

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A, B-Minimal Stirling Numbers

Brian Miceli Trinity University

This talk is based on joint work with Zachary Moring

Given a positive integer n, we denote a set partition of $I_n = \{1, 2, \ldots, n\}$ into k nonempty parts by $P = B_1/B_2/\cdots/B_k$, where $min(B_1) < min(B_2) < \cdots < min(B_k)$. We let $\Pi_{n,k}$ denote the set of all such set partitions, and we define $\min(P) = \{min(B_1), min(B_2), \ldots$ For example, $P = 147/238/56 \in \Pi_{8,3}$, and $\min(P) = \{1, 2, 5\}$. It is well known that $S_{n,k} = |\Pi_{n,k}|$ is the *Stirling number of the second kind*, which for $0 \leq k \leq n$ satisfies the recursion

$$S_{n+1,k} = S_{n,k-1} + kS_{n,k},$$

wth initial conditions $S_{0,0} = 1$ and $S_{n,k} = 0$ if $n < 0$, $k < 0$, or $n < k$.

Over the past few decades, various generalizations of $S_{n,k}$ have appeared in the literature—see Gould [4] for an early example involving q-analogues. While not necessarily explicitly stated in conjunction with any particular generalized Stirling number, there are often implicit conditions being placed on the minimal elements of the parts of the set partition generated by each new type of Stirling number. To this end, given nonnegative integers n, k and disjoints sets $A, B \subseteq I_n$, we define the following two numbers:

$$S_{n,k}(A; B) = |\{P \in \Pi_{n,k} \mid A \subseteq min(\pi) \text{ and } i \notin min(\pi)\ \forall i \in B\}|$$

and

$$\mathcal{B}_n(A; B) = \sum_{k=1}^{n} S_{n,k}(A; B).$$

We call $S_{n,k}(A; B)$ the *A,B-minimal Stirling number of the second kind*, and we thus define $\mathcal{B}_n(A; B)$ to be the *n-th A,B-minimal Bell number*. Setting $S_{n,k}(A; \emptyset) = S_{n,k}(A)$ and $\mathcal{B}_n(A; \emptyset) = \mathcal{B}_n(A)$, we note that $S_{n,k}(\{1\}) = S_{n,k}$, so that $\mathcal{B}_n(\{1\}) = \mathcal{B}_n$, the *n-th (classical) Bell number*. In general we have a multitude of options when choosing A, B, and here we discuss two that relate to interesting mathematical objects:

(i) $S_{n,k}(I_r, \{r+1\}) := S_{n,k,[r]}$, where $\sum_{k=1}^{n} S_{n,k,[r]} := \mathcal{B}_{n,[r]}$, and

(ii) $S_{n,k}(\{i\}; \emptyset) := \tilde{S}_{n,k,\{i\}}$, where $\sum_{k=1}^{n} \tilde{S}_{n,k,\{i\}} := \tilde{\mathcal{B}}_{n,\{i\}}$.

Recall that Broder [2] defined r-Stirling numbers of the second kind, denoted by $\left\{ {n \atop k} \right\}_r$, to count set partitions in which $1, 2, \ldots, r$ are in distinct subsets. This condition, however, is equivalent to simply requiring $I_r \subseteq \min(P)$ for any $P \in \Pi_{n,k}$, that is, $S_{n,k}(I_r) = \left\{ {n \atop k} \right\}_r$. Now, some of the partitions counted by $\left\{ {n \atop k} \right\}_r$ are also counted by $\left\{ {n \atop k} \right\}_{r+1}$. We can slightly alter Broder's numbers by considering the numbers

$S_{n,k,[r]}$, i.e., we enumerate exactly those partitions of I_n such that $1, 2, \ldots, r$ are minimal elements of parts, but $r+1$ is not. The numbers $S_{n,k,[r]}$ are enumerating disjoint sets (unlike Broder's r-Stirling numbers), and so for any r we get $\mathcal{B}_{n,[r]} = \mathcal{B}_n$. More generally, we have the following enumerative result.

Theorem 1. *For* $1 \leq r < n$, $\mathcal{B}_{n,[r]} = r \sum_{i=0}^{n-r-1} \binom{n-r-1}{i} r^i B_{n-r-1-i}$.

Our second generalization, $\tilde{S}_{n,k,\{i\}}$, enumerates exactly those partitions of $[n]$ into k parts such that i is a minimal element of *some* part of our partition. We quickly see that $\tilde{\mathcal{B}}_{n,\{1\}} = \mathcal{B}_n$, and moreover, we have the following theorem.

Theorem 2. *For any positive integer n,* $\sum_{i=1}^{n} \tilde{\mathcal{B}}_{n,\{i\}} = \mathcal{B}_{n+1} - \mathcal{B}_n$.

This result gives an entry point into the realm pattern avoidance in permutations, as both the sequence $\{\mathcal{B}_{n+1} - \mathcal{B}_n\}_{n \geq 1} = 1, 3, 10, 37, 151, \ldots$ and the underlying triangular sequence of $\tilde{S}_{n,k,\{i\}}$'s appear in literature on *generalized* patterns—see [3] and [1], respectively. Indeed, let $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathcal{S}_n$ be a permutation of I_n. We say that σ *avoids* the pattern 1-32 if there do not exist $1 \leq i < j < n$ such that $\sigma_i < \sigma_{j+1} < \sigma_j$, and we define

$$Av_n(\text{1-32}) = \{\sigma \in \mathcal{S}_n \mid \sigma \text{ avoids 1-32}\}.$$

Corollary 3. *For any positive integer n,*

$$\sum_{i=1}^{n} \tilde{\mathcal{B}}_{n,\{i\}} = |\{\sigma \in Av_{n+1}(\text{1-32}) \mid \sigma_n < \sigma_{n+1}\}|.$$

An example is given below, where in the top row we list the elements enumerated by $\sum_{i=1}^{3} \tilde{\mathcal{B}}_{3,\{i\}}$: the element corresponding to i is colored for each $i = 1, 2, 3$. In the bottom row we list out the elements enumerated by $|\{\sigma \in Av_4(\text{1-32}) \mid \sigma_3 < \sigma_4\}|$, where here we have colored the rises at the end of these permutation in order to hint at the fact that the partitions in which i is a minimal element correspond bijectively to permutations where $\sigma_{n+1} = n + 2 - i$.

123	1/23	12/3	13/2	1/2/3	13/2	1/23	1/2/3	12/3	1/2/3
1234	2134	2314	3124	3214	2413	4123	4213	3412	4312

Finally, it is worth noting that there are clearly rook theory interpretations of $S_{n,k}(A; B)$, and also, for different choices of A, B, there are interesting q-analogues of A, B-minimal Stirling numbers.

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Enumeration of Super-Strong Wilf Equivalence Classes of Permutations

Ioannis Michos European University, Cyprus

This talk is based on joint work with Christina Savvidou

Super-strong Wilf equivalence classes in the symmetric group $\mathcal{S}_n$ on n letters were shown in [2] to be in bijection with pyramidal sequences of consecutive differences. In this article we enumerate the latter giving recursive formulae in terms of a two-dimensional analogue of the sequence of non-interval permutations. As a by-product, we give a recursively defined set of representatives of super-strong Wilf equivalence classes in $\mathcal{S}_n$.

Introduction

In this work we continue the study of *super-strong Wilf equivalence* on permutations in n letters that commenced in [2]. This notion was originally referred to as *strong Wilf equivalence* by S. Kitaev et al. in [3]. J. Pantone and V. Vatter in [5] used the term "super-strong Wilf" to distinguish this from a more general notion they defined and called strong Wilf equivalence. Detailed proofs of the results presented here can be found in [4]. Let $\mathbb{P}$ denote the set of positive integers.

Definition 1. ([3, Section 5]) Two words $u, v \in \mathbb{P}^*$ are called *super-strongly Wilf equivalent*, denoted $u \sim_{ss} v$, if there exists a weight-preserving bijection $f : \mathbb{P}^* \to \mathbb{P}^*$ such that $Em(u, w) = Em(v, f(w))$ for all $w \in \mathbb{P}^*$.

In [2] super-strongly Wilf equivalence classes in $\mathcal{S}_n$ were characterized using *sequences of consecutive differences* of permutations. The latter are defined as follows.

Definition 2. ([2, Definition 3]) Let $u \in \mathcal{S}_n$ and $s = s_1 \cdots s_i \cdots s_n = u^{-1}$. For $i = n-1$ down to 1 consider the proper suffix $s_i \cdots s_n$ of s and its alphabet set $\Sigma_i(s) = alph(s_i \cdots s_n) = \{s_i^{(i)}, \ldots, s_n^{(i)}\}$, where $s_i^{(i)} < \cdots < s_n^{(i)}$. We define $\Delta_i(s)$ to be the vector of *consecutive differences* in $\Sigma_i(s)$, i.e., $\Delta_i(s) = (s_{i+1}^{(i)} - s_i^{(i)}, \ldots, s_n^{(i)} - s_{n-1}^{(i)})$. The sequence $p(s) = (\Delta_1(s), \Delta_2(s), \ldots, \Delta_{n-2}(s), \Delta_{n-1}(s))$ has a pyramidal form and is called the *pyramidal sequence of consecutive differences* of $s \in \mathcal{S}_n$.

The main result of [2] is the following.

Theorem 3. *([2, Theorem 3]) Let $u, v \in \mathcal{S}_n$ and $s = u^{-1}, t = v^{-1}$. Then $u \sim_{ss} v$ if and only if $\Delta_i(s) = \Delta_i(t)$, for each $i \in [n-1]$, i.e., if and only if $p(s) = p(t)$.*

To enumerate such pyramidal sequences it is more convenient to leave aside their connections to permutations and focus on the following construction based on three simple rules.

Definition 4. A *pyramidal sequence of vectors* is a sequence of the form

$$p = (\Delta_1, \dots, \Delta_i, \Delta_{i+1}, \dots, \Delta_{n-1}),$$

where each Δ_i is a sequence of $n-i$ positive integers such that $\Delta_1 = (\underbrace{1, 1, \dots, 1}_{n-1})$ and if $\Delta_i = (d_1, d_2, \dots, d_{n-i-1}, d_{n-i})$ we have the following three options for Δ_{i+1}:

$$\Delta_{i+1} = \begin{cases} (d_1, \dots, d_{k-1}, d_k + d_{k+1}, d_{k+2}, \dots, d_{n-i}), \text{ for some } k \in [n-i-1], \text{or} \\ (d_2, \dots, d_{n-i-1}, d_{n-i}), \quad \text{or} \\ (d_1, d_2, \dots, d_{n-i-1}). \end{cases}$$

It is important to note that if $\Delta_i = (\underbrace{d, d, \dots, d}_{n-i})$ for some $d \in \mathbb{P}$, the second and third options coincide. Let Π_n denote the set of all pyramidal sequences of the above form.

It is helpful to view the above definition in the following way. Suppose that we originally have n walls which define $n-1$ chambers with one ball in each one of them. This is precisely the situation in Δ_1. Then at each step the transition from Δ_i to Δ_{i+1} can be visualized by a removal of one wall. If this wall is internal, the balls at its left and right chamber will all be concentrated at one unified chamber. On the other hand, if this wall is external, all corresponding balls to its left (if it is a right wall) or to its right (resp. if it is a left one) will be removed. This combinatorial game ends when all the original $n-1$ balls will be removed. We want to enumerate the number of ways that this can be done, considering that two moves are different if they result to a different set-up of chambers and balls.

Prefixes of generalized non-interval permutations

A word of length $l \geq 2$ is called *periodic* when its vector of consecutive differences is equal to $(\underbrace{d, d, \dots, d}_{l-1})$, for some $d \in \mathbb{P}$.

Definition 5. For $i \in [n-2]$, we define the set $\mathcal{D}_{i,n}$ as the set of words u of length i which appear as non-empty prefixes of permutations in $\mathcal{S}_n$ whose remaining $(n-i)$-lettered suffix is periodic and furthermore this index i is the smallest one attaining that form of periodicity. Set $d_{i,n} = |\mathcal{D}_{i,n}|$.

Definition 6. For $i \in [n-2]$, a *trapezoidal sequence of vectors* is a sequence of the initial parts $(\Delta_1, \Delta_2, \Delta_3, \dots, \Delta_{i+1})$ of an element in Π_n such that $\Delta_{i+1} = (\underbrace{d, d, \dots, d}_{n-i-1})$, for some $d \in \mathbb{P}$ and there is no $j \in [2, i]$ such that $\Delta_j = (\underbrace{e, e, \dots, e}_{n-j})$, for some $e \in \mathbb{P}$. Let $\Delta_{i,n}$ denote the set of all such trapezoidal sequences.

Proposition 7. *Let $n \in \mathbb{P}$ and $i \in [2, n-2]$. There is a bijection between the set of prefixes $\mathcal{D}_{i,n}$ and the set of trapezoidal sequences $\Delta_{i,n}$.*

Sketch of proof. Each word in $\mathcal{D}_{i,n}$ may be viewed as a sequence of i walls that are going to be deleted one after the other, starting from the original n walls containing $n-1$ balls. Taking into consideration the way that the balls are separated after each wall deletion, a unique trapezoidal sequence in $\Delta_{i,n}$ is constructed bottom-up. This is a well defined map and it is not hard to show that it is a bijection. □

The enumeration of super-strong Wilf equivalence classes is then based on the numbers $d_{i,n}$ as follows.

Theorem 8. *The number s_n of distinct super-strong Wilf equivalence classes of $\mathcal{S}_n$ is given by the recursive formula*

$$s_n = s_{n-1} + \sum_{i=2}^{n-2} d_{i,n} \cdot s_{n-i}.$$

Sketch of proof. Let $\mathcal{T}_{i,n} = \{(\Delta_1, \ldots \Delta_i, \Delta_{i+1}, \ldots, \Delta_{n-1}) \in \Pi_n : (\Delta_1, \ldots \Delta_i, \Delta_{i+1}) \in \Delta_{i,n}\}$, for $i \in [n-2]$. We clearly have $\Pi_n = \mathcal{T}_{1,n} \sqcup \mathcal{T}_{2,n} \sqcup \cdots \sqcup \mathcal{T}_{i,n} \sqcup \cdots \sqcup \mathcal{T}_{n-2,n}$. The enumeration of $\mathcal{T}_{i,n}$ is then achieved by observing that any of its pyramidal sequence consists of a trapezoidal part from $\Delta_{i,n}$ and an upper part corresponding to a pyramidal sequence in Π_{n-i} via a suitable scalar factor. □

Corollary 9. *A set of super-strong Wilf equivalence classes representatives in $\mathcal{S}_n$ is described recursively by the set of the inverses of*

$$\mathcal{R}_n = \{u \cdot v \quad : \quad u \in \mathcal{E}_{i,n};\ red(v) \in \mathcal{R}_{n-i};\ i \in [n-2]\},$$

where $red(v)$ is the reduced form of v; $\mathcal{E}_{1,n} = \{1\}$ and $\mathcal{E}_{i,n} = \mathcal{D}_{i,n}$, for $i \geq 2$.

Let $s_{j,n}$ be the number of super-strong Wilf equivalence classes of order 2^j in $\mathcal{S}_n$, where $j \in [n-1]$. Note that $s_{0,n} = 0$.

Theorem 10.

$$s_{j,n} = s_{j-1,n-1} + \sum_{k=2}^{n-j-1} d_{k,n} \cdot s_{j,n-k}.$$

In view of the above results, to calculate s_n and $s_{j,n}$ we need a formula for the coefficients $d_{i,n}$. For given $l, m \in \mathbb{P}$ let $q_{l,m}$ and $r_{l,m}$ be the unique quotient and remainder, respectively, of the Euclidean division of l with m.

Theorem 11. *Let $n \geq 4$. For a given $i \in [n-2]$ set $m = n-i-1$. Then we have the following recursive formula for the numbers $d_{i,n}$*

$$\sum_{k=1}^{i} \frac{q_{n-k,m}}{2} \cdot (r_{n-k,m} + i - k + 1) \cdot d_{k,n} \cdot (i-k)! = \frac{q_{n,m}}{2} \cdot (r_{n,m} + i + 1) \cdot i!.$$

Sketch of proof. Let $p_{i,n}$ be the number of all prefixes u of length i of permutations in $\mathcal{S}_n$ with corresponding suffix v, an m-periodic word. The right hand side of the formula calculates $p_{i,n}$ by counting such periodic words v and multiplying each one of them with the $i!$ choices for u. For the left hand side an alternative counting method is used starting from the prefixes $u' \in \mathcal{D}_{k,n}$ of u. For such a prefix u', the choices for the periodic word v are enumerated likewise and the choices for the suffix of u' in u are $(i-k)!$ due to the remaining $i-k$ letters in u. □

The numbers $d_{i,n}$ are related to the number a_n of *non-interval* permutations, i.e., permutations of size $n \geq 2$ such that any prefix of length $2 \leq l < n$ is not, up to order, equal to the interval $[k, l+k-1]$, for some $k \in [n-l+1]$ [1, Theorem 4.4]. This is the sequence $2, 2, 8, 44, 296, 2312, 20384, \ldots$ (also known as $|b_n|$, where b_n is Sequence A077607 of [6]). It turns out that $d_{k,n} = a_{k+1}$, for $k < \lfloor \frac{n}{2} \rfloor$. This is not a mere coincidence due to the next result.

Proposition 12. *There is a bijection between the set of prefixes $\mathcal{D}_{k,n}$, for $k < \lfloor \frac{n}{2} \rfloor$ and the set $\mathcal{A}_{k+1}$ of all non-interval permutations of length $k+1$.*

$i \backslash n$	3	4	5	6	7	8	9	10	11	12
1	3	2	2	2	2	2	2	2	2	2
2		6	4	2	2	2	2	2	2	2
3			24	16	14	8	8	8	8	8
4				168	100	80	68	44	44	44
5					1,212	712	500	488	416	296
6						10,824	6,376	4,664	3,704	3,512
7							103,992	58,336	43,592	33,152
8								1,114,944	630,544	444,992
9									12,907,824	7,167,802
10										162,773,970

Table 1: The numbers $d_{i,n}$ for $1 \leq i \leq 10$ and $3 \leq n \leq 12$

n	1	2	3	4	5	6	7	8	9	10	11	12
s_n	1	1	2	8	40	256	1,860	15,580	144,812	1,490,564	16,758,972	205,029,338

Table 2: The numbers s_n for $1 \leq n \leq 12$

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Recognizing Merge Classes

Michal Opler Charles University

This talk is based on joint work with Vít Jelínek and Pavel Valtr

We say that a permutation π is a *merge* of σ and τ if the elements of π can be colored red and blue so that the red elements form a copy of σ and the blue elements form a copy of τ. For permutation classes C and D, we denote by $C \odot D$ the class of all possible merges of τ from C and σ from D.

The concept of merges was extensively studied from the structural point of view by Jelínek and Valtr [2]. Here, our goal is to look at this concept from an algorithmic perspective. Let C-recognition be the problem of deciding whether a given permutation π belongs to the permutation class C. We are interested in the computational complexity of $(C \odot D)$-recognition problems for various choices of C and D.

Our first algorithmic approach is based on the concept NLOL-recognizable permutation classes, which we introduce. We say that a class C is *non-deterministically logspace online recognizable*, or NLOL-recognizable, if there is a non-deterministic algorithm A operating in the following way: to recognize whether a given permutation $\pi = \pi_1, \ldots, \pi_n$ is in C, the algorithm A first receives the length n of π, and is given access to $O(\log n)$ bits of memory where it can store arbitrary data. Next, A receives the values $\pi_1, \ldots, \pi_n$ one by one. Upon receiving π_i, A may carry out an arbitrary non-deterministic computation, and then it answers whether the permutation order-isomorphic to $\pi_1, \ldots, \pi_i$ belongs to C. The algorithm can store any data of size $O(\log n)$ in its memory, but it cannot access any part of the input except the latest value π_i that it has received. The algorithm A has to non-deterministically recognize C in the following sense: the sequence $\pi_1, \ldots, \pi_i$ is order-isomorphic to a permutation in C if and only if there is a computation of A over $n, \pi_1, \ldots, \pi_i$ that answers the question positively.

Observe that for a NLOL-recognizable permutation class C, C-recognition can be decided in polynomial time by simply constructing the graph of all possible transitions between memory states during the non-deterministic computation. It can be shown that NLOL, when viewed as a set of permutation classes, is closed under taking unions, intersections, sum closures, gridding, complements, reverses and most importantly merges. Therefore, we obtain the following result.

Theorem 1. *If C and D are NLOL-recognizable permutation classes, then $C \odot D$ is also NLOL-recognizable, and in particular $(C \odot D)$-recognition can be solved in polynomial time.*

This theorem provides non-trivial results, as NLOL contains for example the class of layered permutations, the class of co-layered permutations, the class $\mathrm{Av}(1 \cdots k)$ for any k, or the class of separable permutations with decomposition trees of bounded depth.

Another of our results is based on the concept of *treewidth* of a permutation, defined by Ahal and Rabinovich [1]. We show that for any σ and τ, the $\mathrm{Av}(\sigma) \odot \mathrm{Av}(\tau)$-RECOGNITION problem can be solved in polynomial time on inputs of bounded treewidth. More precisely, we obtain the following result.

Theorem 2. *Let τ and σ be permutations of size at most k, and let π be a permutation of size n and treewidth at most k. Then we can decide whether π belongs to $\mathrm{Av}(\tau) \odot \mathrm{Av}(\sigma)$ in time $O(f(k)n)$ for a function f.*

Finally, we construct polynomial algorithms for several special cases of merges not covered by the previous approaches. For instance, we prove the following result.

Theorem 3. *For $k \geq 0$, $l \geq 2$ and $m \geq 1$, $\mathrm{Av}((1 \cdots k) \oplus (l \dots 1) \oplus (1 \cdots m)) \odot \mathrm{Av}(21)$-RECOGNITION can be solved in polynomial time.*

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On the Growth of Merges and Staircases of Permutation Classes

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This talk is based on joint work with Michael Albert and Vince Vatter

Given two permutation classes $\mathcal{C}$ and $\mathcal{D}$, their *merge*, written $\mathcal{C} \odot \mathcal{D}$, is the set of all permutations whose entries can be colored red and blue so that the red subsequence is order isomorphic to a member of $\mathcal{C}$ and the blue subsequence is order isomorphic to a member of $\mathcal{D}$. We further call such a coloring a $(\mathcal{C}, \mathcal{D})$ *coloring*.

Merges seem to appear only rarely "in nature", with two notable exceptions. First, it is well known that the entries of a $k \cdots 21$-avoiding permutation can be partitioned into $k-1$ increasing subsequences, from which it follows that

$$\mathrm{Av}(k \cdots 21) = \underbrace{\mathrm{Av}(21) \odot \cdots \odot \mathrm{Av}(21)}_{k-1 \text{ copies of } \mathrm{Av}(21)} = \mathrm{Av}((k-1) \cdots 21) \odot \mathrm{Av}(21).$$

First studied by Stankova [13], the class of *skew merged permutations* is also a merge; it is the class $\mathrm{Av}(21) \odot \mathrm{Av}(12)$.

Very little is known about the asymptotic behavior of the sequence $|(\mathcal{C} \odot \mathcal{D})_n|$, even when the asymptotic behaviors of $|\mathcal{C}_n|$ and $|\mathcal{D}_n|$ are known exactly. An upper bound on $|(\mathcal{C} \odot \mathcal{D})_n|$ can be obtained by noting that there are $\binom{n}{i}^2$ ways to partition a permutation of length n into two subpermutations of lengths i and $n-i$, yielding

$$|(\mathcal{C} \odot \mathcal{D})_n| \leq \sum_{i=0}^{n} \binom{n}{i}^2 |\mathcal{C}_i||\mathcal{D}_{n-i}|. \qquad (\dagger)$$

A comparison between (†) and the Binomial Theorem yields the following upper bound on $\overline{\mathrm{gr}}(\mathcal{C} \odot \mathcal{D})$, which first appeared implicitly in the work of Albert [1] and was rediscovered by Claesson, Jelínek, and Steingrímsson [9].

Proposition 1. *For any two permutation classes $\mathcal{C}$ and $\mathcal{D}$,*

$$\overline{\mathrm{gr}}(\mathcal{C} \odot \mathcal{D}) \leq \left(\sqrt{\overline{\mathrm{gr}}(\mathcal{C})} + \sqrt{\overline{\mathrm{gr}}(\mathcal{D})} \right)^2.$$

The only known lower bound on the growth rate of $\mathcal{C} \odot \mathcal{D}$ in general is $\mathrm{gr}(\mathcal{C}) + \mathrm{gr}(\mathcal{D})$, which is achieved by their *juxtaposition*, the class of permutations which consist of a prefix order-isomorphic to a member of $\mathcal{C}$ followed by a suffix order-isomorphic to a member of $\mathcal{D}$. Despite the large gap between bounds, we are not aware of any pair of permutation classes whose merge does *not* achieve the bound in Proposition 1. This is the main question addressed here: Under what conditions on $\mathcal{C}$ and $\mathcal{D}$ can we guarantee that the upper bound on the growth rate of $\mathcal{C} \odot \mathcal{D}$ provided by Proposition 1 is actually achieved?

In the next section we establish a sufficient condition for the growth rate of $\mathcal{C} \odot \mathcal{D}$ to match the upper bound in Proposition 1. A class $\mathcal{C}$ is *sum closed* if $\pi \oplus \sigma \in \mathcal{C}$ for all $\pi, \sigma \in \mathcal{C}$ and *skew closed* if $\pi \ominus \sigma \in \mathcal{C}$ for all $\pi, \sigma \in \mathcal{C}$. A simple application of Fekete's Lemma for super-multiplicative sequences shows that sum closed and skew closed classes have proper growth rates (this argument was first given by Arratia [3]). We can now state our main result.

Theorem 2. *If each of the classes $\mathcal{C}$ and $\mathcal{D}$ is either sum or skew closed then*

$$\mathrm{gr}(\mathcal{C} \odot \mathcal{D}) = \left(\sqrt{\mathrm{gr}(\mathcal{C})} + \sqrt{\mathrm{gr}(\mathcal{D})}\right)^2.$$

In particular, all principal classes are either sum or skew closed, and thus we see that the growth rate of the merge of any two principal classes is equal to the upper bound in Proposition 1.

A striking example of the usefulness of Theorem 2 is its application to $\mathrm{Av}(k \cdots 21)$. Because permutations in this class can be partitioned into $k-1$ increasing subsequences, it is easy—even without appealing to Proposition 1—to see that

$$\mathrm{gr}(\mathrm{Av}(k \cdots 21)) \leq (k-1)^2.$$

That this upper bound is the actual growth rate was first established by Regev [12] via a deep argument (though it should be noted that Regev established quite a bit more as well). However this fact follows easily from Theorem 2 via induction because $\mathrm{Av}(k \cdots 21) = \mathrm{Av}((k-1) \cdots 21) \odot \mathrm{Av}(21)$.

With no known counterexamples, we are compelled to ask if the upper bound on the growth rate of the merge of two classes is always correct:

Question 3. *Is it the case that*

$$\mathrm{gr}(\mathcal{C} \odot \mathcal{D}) = \left(\sqrt{\mathrm{gr}(\mathcal{C})} + \sqrt{\mathrm{gr}(\mathcal{D})}\right)^2$$

for every pair of classes $\mathcal{C}$ and $\mathcal{D}$ with proper growth rates?

We prove Theorem 2 in the next section and present an application of it in a later section. We conclude by discussing a candidate for the "next" most obvious merge to consider in investigating Question 3.

Staircases

In order to prove Theorem 2 we must take a detour to study certain permutation classes that we call *staircase classes*, which are special cases of infinite grid classes of permutations. Therefore, in this section we will first define grid classes and recall an important result about their growth rates. Then, we will define staircase classes,

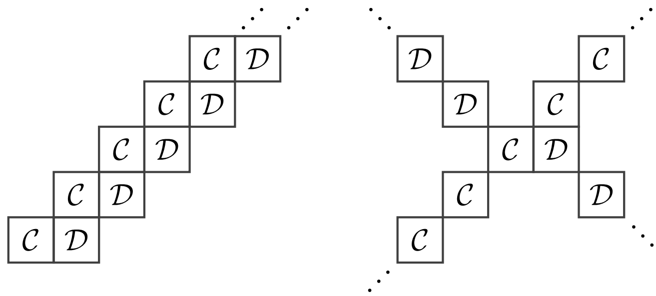

Figure 1: The two staircases we use: the infinite increasing $(\mathcal{C}, \mathcal{D})$ staircase on the left and the infinite counterclockwise spiral $(\mathcal{C}, \mathcal{D})$ staircase on the right.

compute a bound on their growth rates, and conclude the section by illustrating the connection between staircase classes and merges.

Suppose that $\mathcal{M}$ is a $t \times u$ matrix of permutation classes, where t is the number of columns, and u the number of rows. An *$\mathcal{M}$-gridding* of the permutation π of length n is a pair of sequences $1 = c_1 \leq \cdots \leq c_{t+1} = n+1$ (the column divisions) and $1 = r_1 \leq \cdots \leq r_{u+1} = n+1$ (the row divisions) such that for all $1 \leq k \leq t$ and $1 \leq \ell \leq u$, the entries of π with indices in $[c_k, c_{k+1})$ and values in $[r_\ell, r_{\ell+1})$ are order isomorphic to an element of $\mathcal{M}_{k,\ell}$. The *grid class of $\mathcal{M}$*, written $\operatorname{Grid}(\mathcal{M})$, consists of all permutations which possess an $\mathcal{M}$-gridding. The aforementioned juxtaposition of $\mathcal{C}$ and $\mathcal{D}$ can be expressed in this language as $\operatorname{Grid}(\mathcal{C}\ \mathcal{D})$.

By relating their growth rates to the asymptotics of certain walks in a bipartite graph, Bevan [4] gave a formula for growth rates of *monotone grid classes*, that is, those where every cell is either the empty class $\emptyset$, the increasing class $\operatorname{Av}(21)$, or the decreasing class $\operatorname{Av}(12)$. Albert and Vatter [2] have since established the following generalization.

Theorem 4 (Albert and Vatter [2])**.** *Let $\mathcal{M}$ be a $t \times u$ matrix of permutation classes, each with a proper growth rate, and define the $t \times u$ matrix Γ by $\Gamma_{k,\ell} = \sqrt{\operatorname{gr}(\mathcal{M}_{k,\ell})}$. The growth rate of* $\operatorname{Grid}(\mathcal{M})$ *is equal to the greatest eigenvalue of* $\Gamma^T\Gamma$ (*or equivalently, of* $\Gamma\Gamma^T$).

A picture of the infinite increasing $(\mathcal{C}, \mathcal{D})$ staircase is shown on the left of Figure 1. Before defining this staircase as a grid class, we should warn the reader that, so that the entries of our matrices align with those of our permutations, we index matrices in Cartesian coordinates. Thus $\mathcal{M}_{k,\ell}$ denotes the entry in the k^{th} row from the bottom and the ℓ^{th} column from the left. With that warning issued, the infinite increasing $(\mathcal{C}, \mathcal{D})$ staircase is equal to

$$\operatorname{Grid}\begin{pmatrix} & & \iddots & \iddots \\ & \mathcal{C} & \mathcal{D} & \\ \mathcal{C} & \mathcal{D} & & \end{pmatrix}.$$

In our indexing, the entries of the main diagonal of the matrix are equal to $\mathcal{C}$ and the entries of the adjacent diagonal are equal to $\mathcal{D}$.

For the rest of this section we assume that the classes $\mathcal{C}$ and $\mathcal{D}$ both have proper growth rates. We define the *t-step* increasing $(\mathcal{C},\mathcal{D})$ staircase to be the subclass of the infinite staircase corresponding to the first t rows. The matrix defining the t-step increasing $(\mathcal{C},\mathcal{D})$ staircase therefore has t rows and $t+1$ columns. For this grid class, the matrix Γ of Theorem 4 contains diagonal entries equal to $\sqrt{\mathrm{gr}(\mathcal{C})}$ and subdiagonal entries equal to $\sqrt{\mathrm{gr}(\mathcal{D})}$. Furthermore, recalling our unusual matrix indexing, we see that $\Gamma\Gamma^T$ is the $t \times t$ matrix defined by

$$(\Gamma\Gamma^T)_{k,\ell} = \begin{cases} \mathrm{gr}(\mathcal{C}) + \mathrm{gr}(\mathcal{D}) & \text{if } k = \ell, \\ \sqrt{\mathrm{gr}(\mathcal{C})\,\mathrm{gr}(\mathcal{D})} & \text{if } |k-\ell| = 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\Gamma\Gamma^T$ is a tridiagonal Toeplitz matrix, meaning that its nonzero entries are confined to the main diagonal and the two diagonals immediately above and below it (the *tridiagonal* condition) and that its entries along a given diagonal are identical (the *Toeplitz* condition). Tridiagonal Toeplitz matrices are one of the few families of matrices for which exact formulas for their eigenvalues and eigenvectors are known (for example see Meyer [11, Example 7.2.5]); the eigenvalues of a $t \times t$ tridiagonal Toeplitz matrix with subdiagonal entries a, main diagonal entries b, and superdiagonal entries c are given by

$$\lambda_j = b + 2\sqrt{ac}\cos\left(\frac{j\pi}{t+1}\right)$$

for $j = 1,\ldots,t$. Applying this to $\Gamma\Gamma^T$, Theorem 4 implies that the growth rate of any t-step $(\mathcal{C},\mathcal{D})$ staircase is

$$\mathrm{gr}(\mathcal{C}) + 2\sqrt{\mathrm{gr}(\mathcal{C})\,\mathrm{gr}(\mathcal{D})}\cos\left(\frac{1}{t+1}\right) + \mathrm{gr}(\mathcal{D}). \qquad (\ddagger)$$

As $t \to \infty$, the central term approaches $2\sqrt{\mathrm{gr}(\mathcal{C})\,\mathrm{gr}(\mathcal{D})}$, showing that the growth rate of the infinite increasing $(\mathcal{C},\mathcal{D})$ staircase is at least $(\sqrt{\mathrm{gr}(\mathcal{C})} + \sqrt{\mathrm{gr}(\mathcal{D})})^2$.

Proposition 5. *The growth rate of any infinite $(\mathcal{C},\mathcal{D})$ staircase is at least*

$$\left(\sqrt{\mathrm{gr}(\mathcal{C})} + \sqrt{\mathrm{gr}(\mathcal{D})}\right)^2.$$

We are now ready to establish Theorem 2. Suppose that each of $\mathcal{C}$ and $\mathcal{D}$ is either sum closed or skew closed. By symmetry, we may suppose that $\mathcal{C}$ is sum closed. If $\mathcal{D}$ is also sum closed, then we see that every member of the infinite increasing $(\mathcal{C},\mathcal{D})$ staircase is the merge of a permutation from $\mathcal{C}$ and one from $\mathcal{D}$. As the growth rate of this staircase matches the upper bound on the growth rate of $\mathcal{C} \odot \mathcal{D}$ from Proposition 5, we are done. Otherwise, $\mathcal{D}$ must be skew closed. In this case, the members of the infinite counterclockwise spiral $(\mathcal{C},\mathcal{D})$ staircase shown on the right of Figure 1 are contained in $\mathcal{C} \odot \mathcal{D}$, and again we have achieved the upper bound from Proposition 5, completing the proof of Theorem 2.

Growth Rates of Principal Classes

It is somewhat remarkable that the crude bounding below by the containment of staircases and above by the naive merge bound can, in some cases, establish the exact growth rates of classes which are not themselves merges. To give a broad family for which this holds we appeal to a result of Jelínek and Valtr, who investigated the question of which classes are contained in the merge of two proper subclasses. Strengthening earlier results of Bóna [6] and Claesson, Jelínek, and Steingrímsson [9], they established the following (we state their result in a symmetric, skew sum form).

Proposition 6 (Jelínek and Valtr [10]). *For all nonempty permutations α, β, and γ, we have*

$$\mathrm{Av}(\alpha \ominus \beta \ominus \gamma) \subseteq \mathrm{Av}(\alpha \ominus \beta) \odot \mathrm{Av}(\beta \ominus \gamma). \tag{§}$$

We caution the reader that while Theorem 2 gives the growth rate of the class on the right-hand side of (§), this is generally not the growth rate of the class on the left-hand side. Consider, for example, the permutation $4231 = 1 \ominus 12 \ominus 1$, where the class on the right-hand side of (§) has growth rate 16. Indeed, establishing this upper bound of 16 on the growth rate of $\mathrm{Av}(4231)$ was one of the original motivations of Claesson, Jelínek, and Steingrímsson [9]. Bóna [8, 7] has since shown how to further restrict the allowable merges to achieve an upper bound of 13.74.

However, in the case of $\beta = 1$, equality is achieved. Consider the infinite $(\mathrm{Av}(\alpha \ominus 1), \mathrm{Av}(1 \ominus \gamma))$ increasing staircase for any permutations α and γ. Suppose to the contrary that a member π of this staircase were to contain $\alpha \ominus 1 \ominus \gamma$, and consider the position of the entry participating as the '1' between the copies of α and γ. If this entry were to lie in a cell labeled by $\mathrm{Av}(\alpha \ominus 1)$, then there would have to be a copy of α above and to its left, showing that the cell itself contained $\alpha \ominus 1$, a contradiction. Similarly, such an entry cannot lie in a cell labeled by $\mathrm{Av}(1 \ominus \gamma)$, as then it could not contain a copy of γ below and to its right. This shows that $\mathrm{Av}(\alpha \ominus 1 \ominus \gamma)$ contains the infinite $(\mathrm{Av}(\alpha \ominus 1), \mathrm{Av}(1 \ominus \gamma))$ increasing staircase, implying the following result of Bóna.

Theorem 7 (Bóna [6, Theorem 4.2]). *For all permutations α and γ,*

$$\mathrm{gr}(\mathrm{Av}(\alpha \ominus 1 \ominus \gamma)) = \left(\sqrt{\mathrm{gr}(\mathrm{Av}(\alpha \ominus 1))} + \sqrt{\mathrm{gr}(\mathrm{Av}(1 \ominus \gamma))}\right)^2.$$

Our final application of staircases is to establish another result of Bóna.

Theorem 8 (Bóna [5, Theorem 5.5]). *If β is sum indecomposable then*

$$\mathrm{gr}(\mathrm{Av}(1 \ominus \beta)) \geq \left(1 + \sqrt{\mathrm{gr}(\mathrm{Av}(\beta))}\right)^2.$$

Concluding Remarks

In searching for further evidence for, or a counterexample to, Question 3, Theorem 2 shows that at least one of the classes must be neither sum nor skew closed. We must choose at least one of the classes so that it has no proper sum or skew closed subclass with the same growth rate, as otherwise we could use a staircase construction with such subclasses to achieve the upper bound. For this reason we rule out classes such as $\mathrm{Av}(21) \ominus \mathrm{Av}(21)$, which is neither sum nor skew closed but contains a sum closed class, $\mathrm{Av}(21)$, of the same growth rate.

One of the simplest examples of a merge not covered by known results or resolved by the preceding remarks is $\mathrm{Grid}(\mathrm{Av}(21)\ \ \mathrm{Av}(21))$ merged with $\mathrm{Av}(21)$. Here we pose the following instance of Question 3.

Question 9. *Is* $\mathrm{gr}(\mathrm{Grid}(\mathrm{Av}(21)\ \ \mathrm{Av}(21)) \odot \mathrm{Av}(21)) = 3 + 2\sqrt{2}$*?*

Although not relevant to the resolution of Question 9, it is curious that this merge is defined by a finite basis ("most" merges do not seem to be finitely based), in particular,

$$\mathrm{Grid}(\mathrm{Av}(21)\ \ \mathrm{Av}(21)) \odot \mathrm{Av}(21) = \mathrm{Av}(4321, 321654, 421653, 431652, 521643, 531642).$$

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On the Distribution of Peaks (and Other Statistics)

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Peaks

A *peak* in permutation π is a position i where $\pi_{i-1} < \pi_i$ and $\pi_i > \pi_{i+1}$. Let $\mathrm{pk}(\pi)$ be the number of peaks of π. We consider the distribution of pk over various sets of pattern-avoiding permutations. Although $\sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n} q^{\mathrm{pk}(\pi)}z^n$ is complicated, it turns out that $\sum_{\pi\in\mathrm{Av}(B)} q^{\mathrm{pk}(\pi)}z^{|\pi|}$ is particularly nice when $B \subseteq \mathcal{S}_3$.

First, we focus on the case where $|B| = 1$. Since reversing π preserves peaks, we only need to consider permutations avoiding 231, 312, or 321.

The enumeration of peaks over Av(231) is already known and is given in OEIS [4] entry A091894:

$$\sum_{\pi\in\mathrm{Av}(231)} q^{\mathrm{pk}(\pi)}z^{|\pi|} = -\frac{-2zq+2z-1\sqrt{-4z^2q+4z^2-4z+1}}{2zq}.$$

There is a natural bijection between 321-avoiders of length n with k peaks and 312-avoiders of length n with k peaks by preserving left-to-right maxima and rearranging the remaining (smaller) entries. Thus, it remains to compute

$$\sum_{\pi\in\mathrm{Av}(321)} q^{\mathrm{pk}(\pi)}z^{|\pi|} = \sum_{\pi\in\mathrm{Av}(312)} q^{\mathrm{pk}(\pi)}z^{|\pi|}.$$

Baxter [2] used enumeration schemes to compute the initial terms of

$$\sum_{\pi\in\mathrm{Av}(321)} q^{\mathrm{pk}(\pi)}z^{|\pi|},$$

and these terms are given in OEIS [4] entry A236406. However, the bivariate generating function for this distribution is new. In particular, we show that

Theorem 1.

$$\sum_{\pi\in\mathrm{Av}(321)} q^{\mathrm{pk}(\pi)}z^{|\pi|} = -\frac{-2z^3q^2+4z^3q-2z^3-2z^2q+2z^2-1+\sqrt{-4z^2q+4z^2-4z+1}}{2z(zq-z+1)^2}$$

Before we give an outline of the proof of Theorem 1, we consider some statistics on Dyck paths.

Dyck Path Statistics

A Dyck path of semi-length n is a path from $(0,0)$ to $(2n,0)$ that stays at or above the x-axis and uses only the steps $U=(1,1)$ and $D=(1,-1)$. Let $\mathcal{D}_n$ be the set of Dyck paths of semi-length n and $\mathcal{ID}_n$ be the set of indecomposable Dyck paths of semi-length n, where an indecomposable path is one which passes through $(i,0)$ iff $i=0$ or $i=2n$. Let $\mathcal{D}=\cup_{n\geq 0}\mathcal{D}_n$, $\mathcal{ID}=\cup_{n\geq 0}\mathcal{ID}_n$, and $|d|$ be the semi-length of Dyck path d.

We care about two statistics in particular:

1. $\mathrm{st}(d)$ is the number of UUD factors in Dyck path d.
2. $\mathrm{st2}(d)$ is the number of UUD factors in Dyck path d that appear before the last U.

For example,

$$\mathrm{st}(UUUDDDUD)=\mathrm{st2}(UUUDDDUD)=1,$$

while

$$\mathrm{st}(UUUDDDUUDD)=2$$

and

$$\mathrm{st2}(UUUDDDUUDD)=1.$$

It turns out that:

$$A(q,z):=\sum_{d\in\mathcal{D}}q^{\mathrm{st}(d)}z^{|d|}=\sum_{\pi\in\mathrm{Av}(321)}q^{\mathrm{des}(\pi)}z^{|\pi|},$$

$$B(q,z):=\sum_{d\in\mathcal{ID}}q^{\mathrm{st}(d)}z^{|d|}=z(1-q)+\sum_{\pi\in\mathrm{Av}(231)}q^{\mathrm{pk}(\pi)+1}z^{|\pi|+1},$$

$$C(q,z):=\sum_{d\in\mathcal{D}}q^{\mathrm{st2}(d)}z^{|d|}=\sum_{\pi\in\mathrm{Av}(321)}q^{\mathrm{pk}(\pi)}z^{|\pi|},$$

and

$$D(q,z):=\sum_{d\in\mathcal{ID}}q^{\mathrm{st2}(d)}z^{|d|}=\sum_{\pi\in\mathrm{Av}(321)}q^{\mathrm{des}(\pi)}z^{|\pi|+1}.$$

By focusing on the Dyck path statistics, it follows naturally that $A=1+A\cdot B$ and $C=1+A\cdot D$. However, since $D=zA$, we know that $C=1+zA^2$. While this is natural in terms of Dyck paths, it is more interesting in terms of permutations.

In 2010, Barnabei, Bonetti, and Silimbani [1] studied the distribution of descents over $\text{Av}(321)$. They showed

$$A(q,z) = \sum_{\pi \in \text{Av}(321)} q^{\text{des}(\pi)} z^{|\pi|} = -\frac{-1+\sqrt{-4z^2q+4z^2-4z+1}}{2z(zq-z+1)}.$$

From this, we can now derive exact expressions for $B(q,z)$, $C(q,z)$, and $D(q,z)$. In particular, $C(q,z) = 1 + zA(q,z)^2$ is the generating function given in Theorem 1, which is new to the literature.

We expect the distribution of descents and the distribution of peaks over $\text{Av}(321)$ to be related. In particular, since a 321-avoiding permutation has no double descents, every descent is either at the beginning of the permutation or is part of a peak. That is, given $\pi \in \text{Av}_n(321)$, either $\text{des}(\pi) = \text{pk}(\pi)$ or $\text{des}(\pi) = \text{pk}(\pi) + 1$. The fact that the distribution of peaks over $\text{Av}(231)$ arises in conjunction with this same Dyck path statistic is perhaps more surprising.

In [1], Barnabei, Bonetti, and Silimbani used Kratenthaller's bijection between 123-avoiding permutations and Dyck paths [3] to show that the number of descents in a 123-avoiding permutation π is equal to the number of valleys (DU factors) plus the number of triple falls (DDD factors) in the corresponding Dyck path. We now have an even simpler description. By symmetry, 123-avoiding permutations of length n with k descents are in bijection with 321-avoiding permutations of length n with k ascents (i.e. $n-1-k$ descents). We prove the identity

$$\sum_{d \in \mathcal{D}} q^{\text{st}(d)} z^{|d|} = \sum_{\pi \in \text{Av}(321)} q^{\text{des}(\pi)} z^{|\pi|}$$

by letting d_π be the Dyck path corresponding to π under Kratenthaller's bijection and then providing an involution on $\text{Av}_n(321)$ to account for the permutations where $\text{st}(d_\pi) \neq \text{des}(\pi)$. Thus, instead of tracking two statistics on Dyck paths to enumerate the distribution of descents over 321-avoiders, we need only track one statistic (copies of UUD). This map also helps account for the correspondence between the Dyck path interpretation and the permutation interpretation of $D(q,z)$. The correspondence for $C(q,z)$ is simpler to describe, and the correspondence for $B(q,z)$ was already known via recursive descriptions of the appropriate permutations and Dyck paths.

Other Statistics

Beyond peaks, we consider the distribution of double ascents and double descents over $\text{Av}(B)$. In $\text{Av}(231)$ and $\text{Av}(312)$ these statistics correspond to a pattern statistic on the set of all binary trees. In $\text{Av}(321)$, there are no double descents, and the distribution of double ascents is new to the literature.

The distribution of these statistics over $\mathrm{Av}(B)$ where $|B| \geq 2$ results in a number of well-known combinatorial sequences.

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Classical Pattern Distribution in $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$

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This talk is based on joint work with Jeff Remmel

Given a sequence $w = w_1 \dots w_n$ of distinct integers, let red$[w]$ be the permutation found by replacing the i-th smallest integer that appears in σ by i. For example, if $\sigma = 2754$, then red$[\sigma] = 1432$. Given a permutation $\tau = \tau_1 \dots \tau_j$ in the symmetric group S_j, we say that the pattern τ *occurs* in $\sigma = \sigma_1 \dots \sigma_n \in \mathcal{S}_n$ provided there exist $1 \leq i_1 < \cdots < i_j \leq n$ such that red$[\sigma_{i_1} \dots \sigma_{i_j}] = \tau$. We say that a permutation σ *avoids* the pattern τ if τ does not occur in σ. In the theory of permutation patterns, τ is called a *classical pattern*.

Let $\mathcal{S}_n(\tau)$ denote the set of permutations in $\mathcal{S}_n$ which avoid τ. If Λ is a collection of permutations, then we let $\mathcal{S}_n(\Lambda)$ denote the set of permutations in $\mathcal{S}_n$ which avoid each permutation in Λ. Let $\mathrm{occr}_\tau(\sigma)$ denote the number of pattern τ occurrence in the permutation σ. For example, the permutation $\sigma = 867943251$ avoids pattern 132, while it contains pattern 123 and $\mathrm{occr}_{123}(\sigma) = 1$ since the only the subsequence $6, 7, 9$ matches pattern 123.

We have studied distribution of *consecutive pattern*s in 132-avoiding permutations and 123-avoiding permutations in a different research. It is then a natural question to study the distribution of *classical patterns* in 132-avoiding permutations and 123-avoiding permutations.

Given two sets of permutations $\Lambda = \{\lambda_1, \dots, \lambda_r\}$ and $\Gamma = \{\gamma_1, \dots, \gamma_s\}$, we study the distribution of classical patterns $\gamma_1, \dots, \gamma_s$ in $\mathcal{S}_n(\Lambda)$. We define

$$Q^\Gamma_\Lambda(t, x_1, \dots, x_s) = 1 + \sum_{n \geq 1} t^n Q^\Gamma_{n,\Lambda}(x_1, \dots, x_s),$$

where

$$Q^\Gamma_{n,\Lambda}(x_1, \dots, x_s) = \sum_{\sigma \in \mathcal{S}_n(\Lambda)} x_1^{\mathrm{occr}_{\gamma_1}(\sigma)} \cdots x_s^{\mathrm{occr}_{\gamma_s}(\sigma)}.$$

Especially, we have

$$Q^\gamma_\lambda(t, x) = 1 + \sum_{n \geq 1} t^n Q^\gamma_{n,\lambda}(x) \text{ and } Q^\gamma_{n,\lambda}(x) = \sum_{\sigma \in \mathcal{S}_n(\lambda)} x^{\mathrm{occr}_\gamma(\sigma)}.$$

The main goal of our research is to study the generating functions $Q^\gamma_\lambda(t, x)$ when λ and γ are both permutation of length 3. We first study the Wilf-equivalent classes of pattern of length 3 in $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$.

Given a permutation σ, we denote the reverse of σ by σ^r, the complement of σ by σ^c, the reverse-complement of σ by σ^{rc}, and the inverse of σ by σ^{-1}. For example, let $\sigma = 15324$, then $\sigma^r = 42351$, $\sigma^c = 51342$, $\sigma^{rc} = 24315$, $\sigma^{-1} = 14352$.

It is easy to see that $\mathcal{S}_n(123)$ is closed under the operation reverse-complement, and both $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$ are closed under the operation inverse. Thus we have the following lemma.

Lemma 1. *Given any permutation pattern γ,*

$$Q_{123}^{\gamma}(t,x) = Q_{123}^{\gamma^{rc}}(t,x) = Q_{123}^{\gamma^{-1}}(t,x), \quad Q_{132}^{\gamma}(t,x) = Q_{132}^{\gamma^{-1}}(t,x).$$

When we let γ be a pattern of length 3, we can have the following corollary.

Corollary 2. *Considering the distribution of pattern of length* 3*, there are* 4 *Wilf-equivalent classes for* $\mathcal{S}_n(132)$*,*

(1) $Q_{132}^{123}(t,x)$*,* **(2)** $Q_{132}^{213}(t,x)$*,* **(3)** $Q_{132}^{231}(t,x) = Q_{132}^{312}(t,x)$*,* **(4)** $Q_{132}^{321}(t,x)$*,*

and there are 3 *Wilf-equivalent classes for* $\mathcal{S}_n(123)$*,*

(1) $Q_{123}^{132}(t,x) = Q_{123}^{213}(t,x)$*,* **(2)** $Q_{123}^{231}(t,x) = Q_{123}^{312}(t,x)$*,* **(3)** $Q_{123}^{321}(t,x)$*.*

It is easy to check that all the 7 generating functions are different when looking at small cases. Our main goal then becomes studying the 7 generating functions.

Results on generating functions $Q_{\lambda}^{\gamma}(t,x)$

Given $\sigma = \sigma_1 \ldots \sigma_n \in \mathcal{S}_n$, we define $\text{inv}(\sigma) = \left|\{(i,j)|1 \leq i < j \leq n,\ \sigma_i > \sigma_j\}\right|$ to be the number of inversions and $\text{coinv}(\sigma) = \left|\{(i,j)|1 \leq i < j \leq n,\ \sigma_i < \sigma_j\}\right|$ to be the number of coinversions of a permutation σ. In fact, the inversion of a permutation is the same as as the number of occurrence of pattern 21, and the coinversion is the same as the number of occurrence of pattern 12. Clearly, $\text{inv}(\sigma) + \text{coinv}(\sigma) = \binom{n}{2}$. We also let $\text{LRmin}(\sigma)$ be the number of left to right minima of σ.

We use Dyck path bijections to obtain recursions for our generating functions. Given an $n \times n$ square, a Dyck path is a path made up of unit down-steps D and unit right-steps R which starts at $(0,n)$ and ends at $(n,0)$ which stays on or below the diagonal $y = n - x$. Many of our results are proved using a bijection of Krattenthaler [3] $\Phi : S_n(132) \to D_n$ and a bijection of Elizalde and Deutsch [1] $\Psi : \mathcal{S}_n(123) \to \mathcal{D}_n$. The two bijections corresponds statistics like inversions and some classical patterns with statistics on the Dyck paths. Figure 1 shows examples of bijections from $\mathcal{S}_9(132)$ and $\mathcal{S}_9(123)$ to $n \times n$ Dyck path.

We first consider permutations that are avoiding 132 and the distribution of pattern of length 2, i.e. we study the distribution of inv and coinv. We let

$$Q_{n,132}^{12}(q) = \sum_{\sigma \in \mathcal{S}_n(132)} q^{\text{coinv}(\sigma)}, \quad Q_{132}^{12}(t,q) = 1 + \sum_{n \geq 1} t^n Q_{n,132}^{12}(q),$$
$$\text{and} \quad P_n(p,q) = \sum_{\sigma \in \mathcal{S}_n(132)} p^{\text{inv}(\sigma)} q^{\text{coinv}(\sigma)}.$$

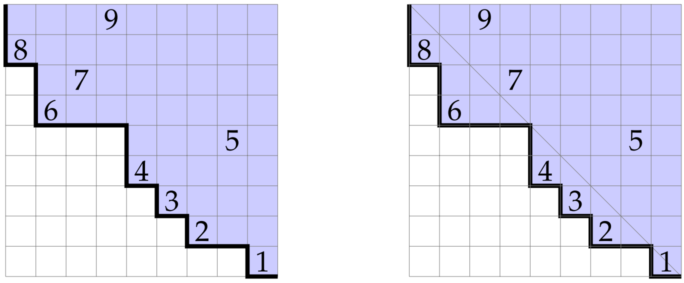

Figure 1: Bijection from $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$ to $n \times n$ Dyck path

Since $\mathrm{inv}(\sigma) + \mathrm{coinv}(\sigma) = \binom{n}{2}$, we have the following relation about $P_n(p,q)$ and $Q_n(q)$,

$$P_n(p,q) = \sum_{\sigma \in \mathcal{S}_n(132)} p^{\binom{n}{2} - \mathrm{coinv}(\sigma)} q^{\mathrm{coinv}(\sigma)} = p^{\binom{n}{2}} Q_n\left(\frac{q}{p}\right).$$

We notice that the statistic inversion of $\sigma \in \mathcal{S}_n(132)$ is equal to the coarea of the corresponding Dyck path. Then, the function $Q^{12}_{n,132}(q)$ is the first type modified q-Catalan number. It is a well known result of Fürlinger and Hofbauer [2] that

Theorem 3 (Fürlinger and Hofbauer). *Let $Q_n(q) = Q^{12}_{n,132}(q)$ and $Q(t,q) = Q^{12}_{132}(t,q)$, then we have the recursions,*

$$Q_0(q) = 1,\ Q_n(q) = \sum_{k=1}^{n} q^{k-1} Q_{k-1}(q) Q_{n-k}(q), \tag{1}$$

$$P_0(q) = 1,\ P_n(q) = \sum_{k=1}^{n} q^{k(n-k)} P_{k-1}(q) P_{n-k}(q), \tag{2}$$

and we have the functional equation,

$$Q(t,q) = 1 + tQ(t,q) \cdot Q(tq,q). \tag{3}$$

We are able to track all patterns of length≤ 3 on $\mathcal{S}_n(132)$ using simple recursion, and we have the following theorem.

Theorem 4. *We let $Q^{\gamma}_{n,132}(q,x) = \sum_{\sigma \in \mathcal{S}_n(132)} q^{\mathrm{coinv}(\sigma)} x^{\mathrm{occr}_{\gamma}(\sigma)}$, then we have the following recursive equations for the generating function $Q^{\gamma}_{n,132}(q,x)$ (we write Q_n for short of $Q^{\gamma}_{n,132}$ on the RHS of each equation).*

$$Q^{\gamma}_{0,132}(q,x) = 1 \quad \text{for each pattern } \gamma, \tag{4}$$

$$Q^{123}_{n,132}(q,x) = \sum_{k=1}^{n} q^{k-1} Q_{k-1}(qx,x) Q_{n-k}(q,x), \tag{5}$$

$$Q^{213}_{n,132}(q,x) = \sum_{k=1}^{n} q^{k-1} x^{\frac{(k-1)(k-2)}{2}} Q_{k-1}(\frac{q}{x},x) Q_{n-k}(q,x), \tag{6}$$

$$Q^{231}_{n,132}(q,x) = \sum_{k=1}^{n} q^{k-1} x^{(k-1)(n-k)} Q_{k-1}(qx^{(n-k)},x) Q_{n-k}(q,x), \tag{7}$$

$$Q^{321}_{n,132}(q,x) = \sum_{k=1}^{n} q^{k-1} x^{\frac{(n-k)(kn-4k+2)}{2}} Q_{k-1}(\frac{q}{x^{n-k}},x) Q_{n-k}(\frac{q}{x^k},x). \tag{8}$$

We can also track all the patterns that

$$\begin{aligned}
& Q_{n,132}^{12,21,123,213,231,312,321}(x_1,x_2,x_3,x_4,x_5,x_6,x_7) \\
= \quad & \sum_{k=1}^{n} x_1^{k-1} x_2^{k(n-k)} x_5^{(k-1)(n-k)} Q_{k-1}(x_1x_3x_5^{(n-k)}, x_2x_4x_7^{(n-k)}, x_3, x_4, x_5, x_6, x_7) \\
& \cdot Q_{n-k}(x_1x_6^k, x_2x_7^k, x_3, x_4, x_5, x_6, x_7). \qquad (9)
\end{aligned}$$

We can compute generating function by Mathematica very fast using the recursion.

n	$Q_{n,132}^{12,21,123,213,231,312,321}(x_1,x_2,x_3,x_4,x_5,x_6,x_7)$
0	1
1	1
2	x_1+x_2
3	$x_1^3x_7+x_1^2x_2x_5+x_1^2x_2x_6+x_1x_2^2x_4+x_2^3x_3$
4	$x_1^6x_7^4+x_1^5x_2x_5^2x_7^2+x_1^5x_2x_5x_6x_7^2+x_1^5x_2x_6^2x_7^2+x_1^4x_2^2x_4x_5^2x_7+x_1^4x_2^2x_4x_6^2x_7+x_1^4x_2^2x_5^2x_6^2$
	$+x_1^3x_2^3x_3x_5^3+x_1^3x_2^3x_3x_6^3+x_1^3x_2^3x_4^3x_7+x_1^2x_2^4x_3x_4^2x_5+x_1^2x_2^4x_3x_4^2x_6+x_1x_2^5x_3^2x_4^2+x_2^6x_3^4$
5	$x_1^{10}x_7^{10}+x_1^9x_2x_5^3x_7^7+x_1^9x_2x_5^2x_6x_7^7+x_1^9x_2x_5x_6^2x_7^7+x_1^9x_2x_6^3x_7^7+x_1^8x_2^2x_4x_5^4x_7^5+x_1^8x_2^2x_4x_5^2x_6^2x_7^5$
	$+x_1^8x_2^2x_4x_6^4x_7^5+x_1^8x_2^2x_5^4x_6^2x_7^4+x_1^8x_2^2x_5^3x_6^3x_7^4+x_1^8x_2^2x_5^2x_6^4x_7^4+x_1^7x_2^3x_3x_5^6x_7^3+x_1^7x_2^3x_3x_5^3x_6^3x_7^3$
	$+x_1^7x_2^3x_3x_6^6x_7^3+x_1^7x_2^3x_4^3x_5^3x_7^4+x_1^7x_2^3x_4^3x_6^3x_7^4+x_1^7x_2^3x_4x_5^4x_6^3x_7^2+x_1^7x_2^3x_4x_5^3x_6^4x_7^2+x_1^6x_2^4x_3x_4^2x_5^5x_7^2$
	$+x_1^6x_2^4x_3x_4^2x_5^4x_6x_7^2+x_1^6x_2^4x_3x_4^2x_5x_6^4x_7^2+x_1^6x_2^4x_3x_4^2x_6^5x_7^2+x_1^6x_2^4x_3x_5^6x_6^3+x_1^6x_2^4x_3x_5^3x_6^6$
	$+x_1^6x_2^4x_4^6x_7^4+x_1^5x_2^5x_3^2x_4^2x_5^5x_7+x_1^5x_2^5x_3^2x_4^2x_6^5x_7+x_1^5x_2^5x_3x_4^5x_5^2x_7^2+x_1^5x_2^5x_3x_4^5x_5x_6x_7^2$
	$+x_1^5x_2^5x_3x_4^5x_6^2x_7^2+x_1^4x_2^6x_3^4x_5^6+x_1^4x_2^6x_3^4x_6^6+x_1^4x_2^6x_3^2x_4^5x_5^2x_7+x_1^4x_2^6x_3^2x_4^5x_6^2x_7+x_1^4x_2^6x_3^2x_4^4x_5^2x_6^2$
	$+x_1^3x_2^7x_3^4x_4^3x_5^3+x_1^3x_2^7x_3^4x_4^3x_6^3+x_1^3x_2^7x_3^3x_4^6x_7+x_1^2x_2^8x_3^5x_4^4x_5+x_1^2x_2^8x_3^5x_4^4x_6+x_1x_2^9x_3^7x_4^3+x_2^{10}x_3^{10}$

Table 1: expression of $Q_{n,132}^{12,21,123,213,231,312,321}(x_1,x_2,x_3,x_4,x_5,x_6,x_7)$

We use the bijection $\Psi : \mathcal{S}_n(123) \to \mathcal{D}_n$ of Elizalde and Deutsch [1] to compute the 3 generation functions for permutations avoiding 123, and we obtained nice recursions. For example, we have

Theorem 5. *Let* $Q_{n,123}^{132}(s,q,x) = \sum_{\sigma\in\mathcal{S}_n(123)} s^{\mathrm{LRmin}(\sigma)} q^{\mathrm{coinv}(\sigma)} x^{\mathrm{occr}_{132}(\sigma)}$, *then we have the following recursions,*

$$Q_{0,123}^{132}(s,q,x) = 1, \quad Q_{n,123}^{132}(s,q,x) = sQ_{n-1} + \sum_{k=2}^{n} Q_{k-1}(sq,qx,x)Q_{n-k}(s,q,x). \qquad (10)$$

From the computation of generating functions, we notice that we can get the recursions and functional equations for the function counting pattern $12\cdots m$ in $\mathcal{S}_n(132)$ and the function counting pattern $1m(m-1)\cdots 2$ in $\mathcal{S}_n(123)$, for any $m > 1$. We found a big coincidence among $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$ that,

$$|\{\sigma \in \mathcal{S}_n(132) : \mathrm{occr}_{12\cdots j}(\sigma) = i\}| = |\{\sigma \in \mathcal{S}_n(123) : \mathrm{occr}_{1j(j-1)\cdots 2}(\sigma) = i\}|$$

for all $i < j$.

This result is described in the following theorem.

Theorem 6. *We let*

$$Q_{n,132}(x_2,x_3,\ldots,x_m) = \sum_{\sigma\in\mathcal{S}_n(132)} x_2^{\mathrm{occr}_{12}(\sigma)} x_3^{\mathrm{occr}_{123}(\sigma)} \cdots x_m^{\mathrm{occr}_{12\cdots m}(\sigma)},$$

$$Q_{132}(t, x_2, x_3, \ldots, x_m) = \sum_{n \geq 0} t^n Q_{n,132}(x_2, x_3, \ldots, x_m) \ \textit{and}$$

$$Q_{n,123}(s, x_2, x_3, \ldots, x_m) = \sum_{\sigma \in \mathcal{S}_n(123)} s^{\mathrm{LRmin}(\sigma)} x_2^{\mathrm{occr}_{12}(\sigma)} x_3^{\mathrm{occr}_{132}(\sigma)} \cdots x_m^{\mathrm{occr}_{1m(m-1)\cdots 2}(\sigma)},$$

$$Q_{123}(t, s, x_2, x_3, \ldots, x_m) = \sum_{n \geq 0} t^n Q_{n,123}(s, x_2, x_3, \ldots, x_m),$$

then we have the following equations,

$$\begin{aligned}
Q_{n,132}(x_2, \ldots, x_m) &= \sum_{k=1}^{n} x_2^{k-1} Q_{k-1,132}(x_2x_3, x_3x_4, \ldots, x_{m-1}x_m, x_m) Q_{n-k,132}(x_2, \ldots, x_m), && (11) \\
Q_{n,123}(s, x_2, \ldots, x_m) &= sQ_{n-1,123}(t, s, x_2, \ldots, x_m) && (12) \\
&\quad + \sum_{k=2}^{n} Q_{k-1,123}(sx_2, x_2x_3, x_3x_4, \ldots, x_{m-1}x_m, x_m) Q_{n-k,123}(s, x_2, \ldots, x_m), \\
Q_{132}(t, x_2, \ldots, x_m) &= 1 + Q_{132}(tx_2, x_2x_3, x_3x_4, \ldots, x_{m-1}x_m, x_m) Q_{132}(t, x_2, \ldots, x_m), && (13) \\
Q_{123}(t, s, x_2, \ldots, x_m) &= 1 + t(s-1) Q_{123}(t, s, x_2, \ldots, x_m) \\
&\quad + tQ_{123}(t, sx_2, x_2x_3, x_3x_4, \ldots, x_{m-1}x_m, x_m) Q_{123}(s, x_2, \ldots, x_m). && (14)
\end{aligned}$$

Further, let $[x^i]_Q$ denote the coefficient of x^i in function Q, then

$$[t^n x_i^j]_{Q_{132}} = [t^n x_j^i]_{Q_{123}} \ \text{ for } \ i < j. \tag{15}$$

Equation (11) and (12) give recursions for functions $Q_{n,132}$ and $Q_{n,123}$, and Equation (13) and (14) give functional equations that Q_{132} and Q_{123} satisfies.

Other generalizations and future work

We obtained the recursion tracking all patterns of length≤ 4 on $\mathcal{S}_n(132)$ in the way of equation (9), and we see that every pattern is trackable by recursion on $\mathcal{S}_n(132)$.

On $\mathcal{S}_{123}$, we only track patterns of length 2 and 3 and the special pattern $1m(m-1)\cdots 2$. The recursions on $\mathcal{S}_n(123)$ tend to be more complicated than those on $\mathcal{S}_n(132)$, thus a simpler recursion is on $\mathcal{S}_n(123)$ desired.

We then adapt our method to circular permutations. We define patters on circular permutations, and we can track all circular patterns of size≤ 4 on circular permutations avoiding circular pattern 1243.

There are other equality of coefficients of generating functions Q_{132}^{γ} and Q_{123}^{γ} except equation (15) which we can study in the future. We did not study sets of permutations avoiding patterns other than 123 and 132, and circular patterns other than 1243.

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On Two-Sided Gamma-Positivity for Simple Permutations

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This talk is based on joint work with Ron M. Adin, Eli Bagno, and Estrella Eisenberg

Introduction

Eulerian numbers enumerate permutations according to their descent numbers. The *two-sided Eulerian numbers*, studied by Carlitz, Roselle, and Scoville [7] constitute a natural generalization. These numbers count permutations according to the number of their descents as well as the number of descents of the inverse permutation.

Definition 1. A polynomial $f(q)$ is *palindromic* if its coefficients are the same when read from left to right as from right to left. Explicitly, if $f(q) = a_r q^r + a_{r+1} q^{r+1} + \cdots + a_s q^s$ with $a_r, a_s \neq 0$ and $r \leq s$, then we require $a_{r+i} = a_{s-i}$ $(\forall i)$; equivalently, $f(q) = q^{r+s} f(1/q)$.

Following Zeilberger [17], we define the *darga* of $f(q)$ as above to be $r+s$; the zero polynomial is considered palindromic of each nonnegative darga. The set of palindromic polynomials of darga n is a vector space of dimension $\lfloor n/2 \rfloor + 1$, with *gamma basis* $\Gamma_n = \{q^j (1+q)^{n-2j} \mid 0 \leq j \leq \lfloor n/2 \rfloor\}$. The (one-sided) *Eulerian polynomial*

$$A_n(q) = \sum_{\pi \in S_n} q^{\mathrm{des}(\pi)}$$

is palindromic of darga $n-1$, and thus there are real numbers $\gamma_{n,j}$ such that

$$A_n(q) = \sum_{0 \leq j \leq \lfloor (n-1)/2 \rfloor} \gamma_{n,j} q^j (1+q)^{n-1-2j}.$$

See [13, p. 72, 78] for details.

Foata and Schützenberger [8] proved that the coefficients $\gamma_{n,j}$ are actually non-negative integers. The result of Foata and Schützenberger was reproved combinatorially, using an action of the group $\mathbb{Z}_2^n$ on S_n which realizes each coefficient $\gamma_{n,j}$ as the number of orbits of a certain type. This method, called “Valley hopping", is described in [9, 5]. A nice exposition appears in [12].

Now let $A_n(s,t)$ be the *two-sided Eulerian polynomial*

$$A_n(s,t) = \sum_{\pi \in S_n} s^{\mathrm{des}(\pi)} t^{\mathrm{ides}(\pi)},$$

where $ides(\pi) = des(\pi^{-1})$.

It is well known (see, e.g., [12, p. 167]) that the bivariate polynomial $A_n(s,t)$ satisfies

$$A_n(s,t) = (st)^{n-1} A_n(1/s, 1/t)$$

and

$$A_n(s,t) = A_n(t,s).$$

A bivariate polynomial satisfying these equations will be called *palindromic of darga* $n-1$.

It can be proved (see [13, p. 78]) that any bivariate palindromic polynomial of darga $n-1$ can be written uniquely in the basis $\{(st)^i(s+t)^j(1+st)^{n-1-j-2i} \mid i,j \geq 0, 2i+j \leq n-1\}$. Gessel (see [5, Conjecture 10.2]) conjectured that the coefficients of $A_n(s,t)$ in that basis are nonnegative integers. This has recently been proved by Lin [11]. Explicitly:

Theorem 2. (Gessel's conjecture, Lin's theorem) *For any* $n \geq 1$ *the bivariate polynomial* $A_n(s,t)$ *is gamma-positive, i.e., there exist nonnegative integers* $\gamma_{n,i,j}$ $(i,j \geq 0, 2i+j \leq n-1)$ *such that*

$$A_n(s,t) = \sum_{i,j} \gamma_{n,i,j}(st)^i(s+t)^j(1+st)^{n-1-j-2i}.$$

Unlike the proof for the univariate version, Lin's proof is not combinatorial.

We propose here an analogous (and stronger) conjecture for the class of *simple permutations*.

Conjecture 3. *For each* $n \geq 2$,

$$\mathrm{simp}_n(s,t) = \sum_{\sigma \in \mathrm{Simp}_n} s^{\mathrm{des}(\sigma)} t^{\mathrm{ides}(\sigma)}$$

is gamma-positive, where Simp_n *is the set of simple permutations of length n.*

Using the substitution decomposition tree of a permutation (by repeated inflation), we show that this would imply the original Gessel-Lin result. We also provide supporting evidence for this conjecture.

This is an extended abstract. More details and proofs may be found in the full version [1].

Preliminaries: notations and definitions

We start with the following:

Observation 4. Let $\pi[\alpha_1, \ldots, \alpha_k]$ be the inflation of the permutation π by $\alpha_1, \ldots, \alpha_k$.

Then $\mathrm{des}(\sigma) = \mathrm{des}(\pi) + \sum_{i=1}^{n} \mathrm{des}(\alpha_i)$ and $\mathrm{ides}(\sigma) = \mathrm{ides}(\pi) + \sum_{i=1}^{n} \mathrm{ides}(\alpha_i)$.

The following proposition claims that every permutation has a canonical representation as an inflation of a simple permutation.

Proposition 5. *[4] Let $\sigma \in S_n$, $(n \geq 2)$. There is a unique simple permutation $\pi \neq 1$ and a sequence of permutations $\alpha_1, \ldots, \alpha_k$ such that $\sigma = \pi[\alpha_1, \ldots, \alpha_k]$.*

If $\pi \notin \{12, 21\}$ then $\alpha_1, \ldots, \alpha_k$ are uniquely determined by σ. If $\pi = 12$ or $\pi = 21$ then α_1, α_2 are unique as long as we require that α_2 is sum-indecomposable or skew-indecomposable respectively.

Example 6. Consider the permutation $\sigma = 452398167$. We first write σ as an inflation of 2413 as follows: $\sigma = 2413[3412, 21, 1, 12]$.

One can continue the process of inflation also for the α_i in a recursive way, until all the permutations participating are simple. In our example, 3412 can be decomposed further as $3412 = 21[12, 12]$ so after substituting in σ we get $\sigma = 2413[21[12, 12], 21, 1, 12]$.

Remark 7. In order to clarify why the requirement that α_2 is sum-indecomposable in inflating 12 is necessary, note that the permutation 123 can be written as $12[1, 12] = 1\ 23$ or as $12[12, 1] = 12\ 3$. The second alternative is the one of them we choose.

This information can be easily described by a tree. We represent each $\pi \in S_n$ by a tree with n leaves. For each permutation $\sigma = \pi[\alpha_1, \ldots, \alpha_n]$, the simple permutation π is represented by the root node and each α_i contributes a child of the root which is an (unlabeled) leaf if $\alpha_i = 1$ and a root of a sub-tree otherwise. In the latter case, if α_i is not simple then by Proposition 5, α_i can be represented as $\alpha_i = \pi'[\beta_1, \ldots, \beta_k]$ with π' simple. We take π' to be the root of the new sub-tree. If α_i is simple of length k then we can write $\alpha_i = \alpha_i[1, 1, \ldots, 1]$ and it has k (unlabeled) leaves. We proceed in a recursive manner.

Definition 8. For $\sigma \in S_n$ denote by T_σ the tree constructed by this process.

Example 9. Figure 1 depicts the trees of $\sigma_1 = 6713245, \sigma_2 = 1257634$.

Inflation is actually a localized version of the *wreath product* construction introduced in [4]

Definition 10. Let $\mathcal{A}, \mathcal{B}$ be sets of permutations. The *wreath product* of $\mathcal{A}$ and $\mathcal{B}$ is $\mathcal{A} \wr \mathcal{B} = \{\alpha[\beta_1, \cdots, \beta_n] \mid \alpha \in \mathcal{A}, \beta_1, \beta_2, \ldots, \beta_n \in \mathcal{B}\}$.

Example 11. Let $A = \{12\}$ and $B = \{21, 132\}$, then $A \wr B = \{2143, 21354, 13254, 132465\}$.

We say that a set H of permutations is *wreath-closed* if $H = H \wr H$. The wreath closure $\mathrm{wc}(H)$ of a set H of permutations, is the smallest wreath-closed set of permutations that contains H. The wreath product operation is associative and so, if we define $H_1 = H$ and $H_{n+1} = H \wr H_n$ then $\mathrm{wc}(H) = \bigcup_{n=1}^{\infty} H_n$.

Example 12. Let $A = \{21\}$. It is easy to see that $A \wr A = \{4321\}$ and $(A \wr A) \wr A = \{87654321\}$ and continuing in this manner one obtains that $\mathrm{wc}(A) = \{[2^n, 2^n - 1, \cdots, 1] \mid n \in \mathbb{N}\}$.

Finally, we have the following definition:

Definition 13. Let Simp_n ($\mathrm{Simp}_{\leq n}$) be the set of all simple permutations of length n (respectively, less or equal to n) respectively and denote by $H(n)$ the wreath-closure of $\mathrm{Simp}_{\leq n}$, i.e. $H(n) = \mathrm{wc}(\mathrm{Simp}_{\leq n})$. Furthermore, let $\mathrm{simp}_n(s,t) = \sum_{\sigma \in \mathrm{Simp}_n} s^{\mathrm{des}(\sigma)} t^{\mathrm{ides}(\sigma)}$

Example 14. $H(2) = \mathrm{wc}\{1, 12, 21\}$ is the set of all permutations that can be obtained from the trivial permutation 1 by direct sums and skew sums. These are the *separable permutations*, counted by *large Schröder numbers* (see [16]). The class of separable permutations can also be written using pattern avoidance: $H(2) = \mathrm{Av}(3142, 2413)$. For more details see [6], right after Proposition 3.2.

Two-sided gamma-positivity: partial results

Lin's theorem (Gessel's conjecture) for the case $H(2) \cap S_n$ can be derived from the work of Fu, Lin and Zeng [10], even though they only dealt with the univariate case. This is due to the fact that $des(\pi) = ides(\pi)$ for any $\pi \in H(2)$. Explicitly:

Theorem 15. *For each n there exist non-negative integers* $\gamma_{n,k}$ $(0 \leq k \leq \lfloor (n-1)/2 \rfloor)$ *such that:*

$$\sum_{\pi \in H(2) \cap S_n} s^{\mathrm{des}(\pi)} t^{\mathrm{ides}(\pi)} = \sum_{\pi \in H(2) \cap S_n} (st)^{\mathrm{des}(\pi)} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,k} (st)^k (1+st)^{n-1-2k}.$$

Every root which has a simple permutation of length 2 as a label (i.e. 12 or 21) is a root of a binary sub-tree. We give special attention to the maximal binary sub-trees. For each maximal binary sub-tree, following [6], we define the notion of a right chain as follows:

In order to further generalize Theorem 15, we have to introduce some more definitions.

Definition 16. Let T be a tree. A *binary right chain* (BRC) is a maximal chain composed of consecutive right descendants, all of which are from the set $\{12, 21\}$.

The *length* of a BRC is the number of nodes in this chain. Note that a binary right chain may also contain only one node. We denote by $r_{odd}(T)$ the number of BRC in T of odd length.

Definition 17. A tree T is called a G-tree if it is labeled by simple permutations and the following "right chain condition" is satisfied i.e. the labeling on each BRC alternates between 12 and 21. Denote by $\mathcal{GT}_n$ the set of all G-trees which have exactly n leaves. (Recall that each S_n permutation can be represented as such a tree).

Note that the right chain condition corresponds precisely to the way we constructed the inflation in Remark 7. By the preceding argument, we can deduce the following theorem:

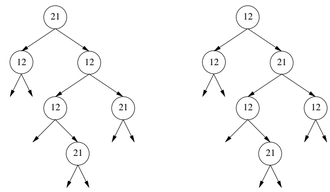

Figure 1: Right: the tree T. Left: The tree $\phi_1(T)$.

Theorem 18. *For each k, the function which sends each $\pi \in H(k) \cap S_n$ to the G-tree T_π from Definition 8 is an injection.*

We present now a combinatorial proof for Gessel's conjecture for $H(5) \cap S_n$, where n is arbitrary. Until the end of this section, all permutations are assumed to belong to $H(5) \cap S_n$.

Let $T = T_\pi$ be a G-tree, and let $\{R_i \mid 1 \leq i \leq r_{odd}(T)\}$ be the set of all BRC of odd length in T. For each i, denote by ϕ_i the operator which switches 12 with 21 in each of the nodes of R_i. By abuse of notation, denote by $\phi_i(T)$ the tree obtained from T in this fashion [1]. It is easy to see that the operators ϕ_i commute and, by Observation 4, each application of any ϕ_i changes both $\text{des}(\pi)$ and $\text{ides}(\pi)$ by ± 1.

Example 19. Consider $\pi = 1257634$. The corresponding G-tree appears in the right side of Fig 1. The tree $T = T_\pi$ has $r_{odd}(T) = 2$. If we number the odd chains from right to left, then we have that $\phi_1(T)$ is the tree in the left side of the figure. The permutation corresponding to $\phi_1(T)$ is 6713254. Note that ϕ_1 increased both $\text{des}(\pi)$ and $\text{ides}(\pi)$ by 1.

For each G-tree T and the corresponding permutation π, let $l_1, \ldots, l_k$ be the labels of T that belong to the set $\{2413, 3142\}$. For each $1 \leq j \leq k$ define $\psi_j(T)$ to be the tree obtained from T by changing the label l_j from 2413 to 3142 or vice versa. Again, it is easy to see that all the ψ_j commute. Passing from 2413 to 3142 adds one to the $\text{des}(\pi)$ and subtracts 1 from $\text{ides}(\pi)$. For any two G-trees T, S write $T \sim S$ if S can be obtained from T by a sequence of applications of the actions ϕ_i and ψ_j we have just defined. Clearly $\sim$ is an equivalence relation, partitioning the set $\mathcal{GT}_n$ and hence also S_n into equivalence classes.

Definition 20. For each class in $\mathcal{GT}_n$ we define the representative of the class to be the tree T_0 in which each odd BRC begins with 12 and each node representing a simple permutation of length 4 is 2413. It is clear from the construction that the corresponding permutation, π_0, has the minimal number of descents in its class. We refer to both as the *minimal representatives* of the class.

Our next goal is to compute the bi-distribution of des and ides over an equivalence class. In order to do that, we pick the minimal representative T_0 and use the actions ϕ_i and ψ_j to go over the entire class.

[1] This action is inspired by a similar action defined in [10] for univariate polynomials.

Theorem 21. *Let $n \in \mathbb{N}$. There exist nonnegative integers $\alpha_{n,i,j}$ $(1 \le i \le n, 0 \le j \le n-1, 0 \le j+2i \le n-1)$ such that:*

$$\sum_{\sigma \in H(5) \cap S_n} s^{des(\sigma)} t^{ides(\sigma)} = \sum_{i,j} \alpha_{n,i,j} (st)^i (s+t)^j (1+st)^{n-1-2i-j}.$$

The BI-Eulerian polynomial for simple permutations

In [3], the ordinary generating function for the number of simple permutations was shown to be very close to the functional inverse of the corresponding generating function for all permutations. In this section we refine this result by considering also the parameters des and ides. For each $n \in \mathbb{N}$, denote by I_n^+ (I_n^-) the set of all sum-indecomposable (respectively, skew-indecomposable) permutations in S_n. Note that the map $\pi \mapsto \pi'$, defined by $\pi'(i) = n - 1 - \pi(i)$ $(1 \le i \le n)$, is a bijection from S_n onto itself (and also from I_n^+ onto I_n^-), satisfying $\mathrm{des}(\pi') = n - 1 - \mathrm{des}(\pi)$ and $\mathrm{ides}(\pi') = n - 1 - \mathrm{ides}(\pi)$. Using some further manipulations we get:

Theorem 22. *For any $n \ge 4$*

$$\mathrm{simp}_n(s,t) = -f_n^{\langle -1 \rangle}(s,t) + (-1)^{n-1} + (-st)^{n-1}.$$

A two-sided gamma-positivity conjecture for simple permutations

In this final section, we present a two-sided gamma-positivity conjecture for simple permutations, and show that it implies Lin's theorem (Gessel's conjecture). Recall that Simp_n is the set of all simple permutations in S_n. Since Simp_n is closed under the actions of reverse and inverse, we have by arguments of symmetry (see [12, p. 78]) the following:

Proposition 23. *Let $n \in \mathbb{N}$. There exist $\alpha_{n,i,j} \in \mathbb{R}$, $(i,j \ge 0,\ j+2i \le n-1)$ such that:*

$$\mathrm{simp}_n(s,t) = \sum_{\pi \in \mathrm{Simp}_n} s^{\mathrm{des}(\pi)} t^{\mathrm{ides}(\pi)} = \sum_{i,j} \alpha_{n,i,j} (st)^i (s+t)^j (1+st)^{n-1-2i-j}.$$

In a previous section we defined actions on the set of simple permutations for $n = 2$ (the reverse action), $n = 4$ (the reverse action again), and also $n = 5$ (the identity action). It follows that $\mathrm{simp}_2(s,t) = 1 + st$, $\mathrm{simp}_4(s,t) = st(s+t)$, and $\mathrm{simp}_5(s,t) = 6(st)^2$.

These techniques do not work in the case of $n = 6$. For example, the simple permutation 246135 contributes st^3 while its reverse contributes s^4t^2. On the other hand, one can easily check that summing over all the simple permutations of order 6 gives a gamma-positive polynomial: $\mathrm{simp}_6(s,t) = st(s+t)^2(1+st) + 5(st)^2(1+st) + 14(st)^2(s+t)$.

In fact, computer experiments show that $\mathrm{simp}_n(s,t)$ are gamma-positive at least for $n \le 12$. We therefore propose the following conjecture:

Conjecture 24. The bivariate polynomial $\text{simp}_n(s,t)$ is gamma-positive for every $n \geq 2$, i.e., there exist nonnegative integers $\alpha_{n,i,j}$ $(0 \leq j + 2i \leq n-1)$ such that:

$$\text{simp}_n(s,t) = \sum_{\pi \in \text{Simp}_n} s^{\text{des}(\pi)} t^{\text{ides}(\pi)} = \sum_{i,j} \alpha_{n,i,j} (st)^i (s+t)^j (1+st)^{n-1-2i-j}.$$

If the conjecture is true then we can expand Theorem 21 to S_n in the following way: First, for each $k \geq 6$, since we still have no action on the simple permutations of S_k, for each $\sigma = \pi[\alpha_1, \ldots, \alpha_t] \in S_n$, we construct a tree having π as a label of its root, in the same way which was explained in the body of the paper. In this general case, however, the nodes are labeled by simple permutations of arbitrary length. Then we define an equivalence relation on S_n by declaring that two trees belong to the same class if one can be reached from the other by applying a series of actions of the following two types: (1).Replace a label of one node (which is a simple permutation of length $k \geq 4$) by a simple permutation of the same order. (2) Alternate the labels of an odd right chain, as described in the paragraph after Theorem 18. Now, given a permutation σ and its tree T_σ, the polynomial $S_{[\sigma]} = \sum_{\tau \sim \sigma} s^{\text{des}(\tau)} t^{\text{ides}(\tau)}$ is a multiplication of polynomials corresponding to the nodes as follows:

(1) Each node representing a simple permutation of length $k \geq 4$ contributes a term of the form $\text{simp}_k(s,t)$, of darga $k-1$.

(2) The v_2 nodes of T_σ of length 2 contribute $(st)^{\frac{v_2 - r_{odd}(T_\sigma)}{2}} (1+st)^{r_{odd}(T_\sigma)}$, of darga v_2.

In summary, for each i, let l_i be the length of the simple permutation labeling the node v_i. Assume that T_σ has m nodes. Then each node v_i contributes $l_i - 1$ to the darga of $S_{[\sigma]}$. As a result we obtain that the darga of $S_{[\sigma]}$ is $\sum_{i=1}^{m} (l_i - 1) = n - 1$.

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On the Distribution of Statistics for Pattern Avoiding Permutations

Jacob Roth Valparaiso University

This talk is based on joint work with Michael Bukata, Ryan Kulwicki, Nicholas Lewandowski, and Teresa Wheeland

Simion and Schmidt [1] enumerated $S_n(\rho_1, \rho_2)$ where $\rho_1, \rho_2 \in S_3$. This poster highlights our examination of the distribution of ascents, descents, double ascents, double descents, peaks, and valleys over these pattern classes. By the Erdős-Szekeres theorem, $S_n(123, 321) = \emptyset$ for $n \geq 5$, so we consider the other 5 trivial Wilf classes below. For sufficiently large n, each formula will give the number of permutations of length n with the given statistic equal to k. Also, A076791 is the entry in The On-Line Encyclopedia of Integer Sequences [2] regarding the number of binary sequences of length n containing k 00 subsequences. We provide combinatorial proofs for each of these distributions.

Patterns	Ascents	Descents	Double Ascents	Double Descents	Peaks	Valleys
123,132	$\binom{n}{2k}$	$\binom{n}{2(n-k-1)}$	trivial	$\binom{n-2}{k} + 2\binom{n-3}{k}$	$\binom{n}{2k+1}$	$2 \cdot \binom{n-1}{2k}$
132,213	$\binom{n-1}{k}$	$\binom{n-1}{k}$	A076791	A076791	$\binom{n}{2k+1}$	$\binom{n}{2k+1}$
132,321	$1, \quad k = n-1$ $\binom{n}{2}, \quad k = n-2$	$1, \quad k = 0$ $\binom{n}{2}, \quad k = 1$	$1, \quad k = n-2$ $n, \quad k = n-3$ $\binom{n}{2} - n, \quad k = n-4$	trivial	$n, \quad k = 0$ $\binom{n-1}{2}, \quad k = 1$	$2, \quad k = 0$ $\binom{n}{2} - 1, \quad k = 1$
213,231	$\binom{n-1}{k}$	$\binom{n-1}{k}$	A076791	A076791	$\binom{n}{2k+1}$	$\binom{n}{2k+1}$
213,312	$\binom{n-1}{k}$	$\binom{n-1}{k}$	$n, \quad k = 0$ $\binom{n-1}{k+1}, \quad k \geq 1$	$n, \quad k = 0$ $\binom{n-1}{k+1}, \quad k \geq 1$	$2, \quad k = 0$ $2^{n-1} - 2, \quad k = 1$	trivial

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Stack Sorting Tiers

Howard Skogman SUNY Brockport

This talk is based on joint work with Toufik Mansour and Rebecca Smith

Knuth [1] showed that a permutation π can be sorted by a stack (meaning that by applying push and pop operations to the sequence of entries $\pi(1), \ldots, \pi(n)$ we can output the sequence $1, \ldots, n$) if and only if π avoids the permutation 231, i.e., if and only if there do not exist three indices $1 \le i_1 < i_2 < i_3 \le n$ such that $\pi(i_1), \pi(i_2), \pi(i_3)$ are in the same relative order as 231.

We consider the number of *passes* a permutation needs to take through a stack if we only pop the appropriate output values and start over with the remaining entries. We define a permutation π to be k-pass sortable if π is sortable using k passes through the stack where the remaining entries are passed through the stack in their original order on later attempts. Permutations that are 1-pass sortable are simply the stack sortable permutations as defined by Knuth.

Example 1. We show the sorting of $\pi = 356124$ using three passes through a stack in Figure 1.

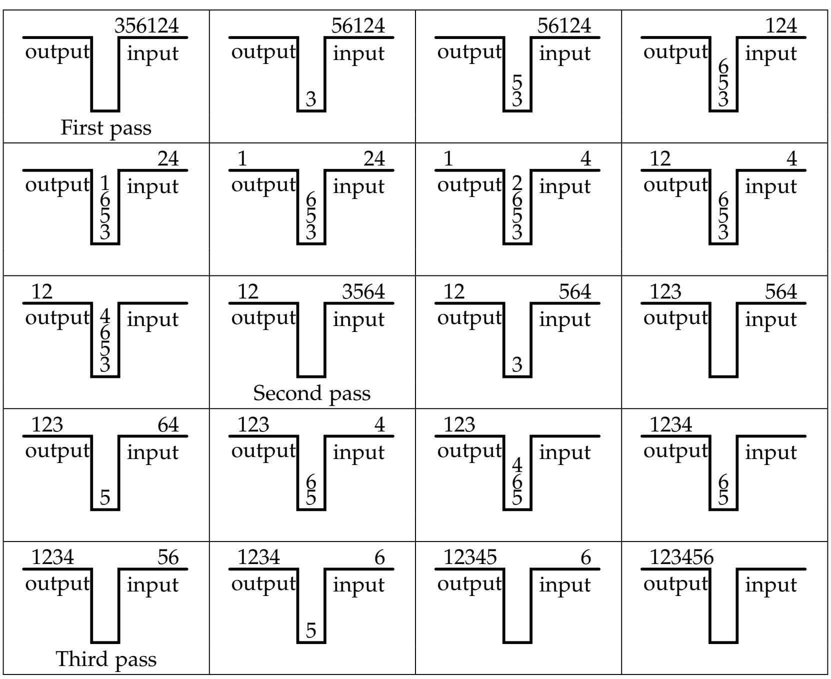

Figure 1: Sorting the permutation 356124 with $k = 3$ passes through a stack.

The *tier* of a permutation π, denoted $t(\pi)$, will be defined to be the minimum number of passes *after* the first pass required to sort π. We then introduce the notion of a separated pair, which is a specific covincular 231 pattern. One can determine the tier of a permutation by computing the number of such separate pairs.

Theorem 2. *A permutation π is 2-pass sortable, i.e. $t(\pi) \leq 1$, if and only if π avoids*

$$24153, 24513, 24531, 34251, 35241, 42513, 42531, 45231, 261453, 231564, 523164.$$

The class of permutations of tier t is in bijection with a collection of integer sequences studied by Parker [2]. This gives an exact enumeration of tier t permutations of a given length and thus an exact enumeration for the class of $(t+1)$-pass sortable permutations. This also allows us to give a new derivation for the generating function in [2] and an explicit formula for the coefficients.

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Some $1 \times n$ Generalized Grid Classes Are Context-Free

Jakub Sliačan Umeå University

This talk is based on joint work with Robert Brignall

This work is concerned with the study of $1 \times n$ almost-monotone permutation grid classes. In particular, we prove that $1 \times n$ grid classes with $n - 1$ monotone cells and one context-free cell (see Figure 1) admit algebraic generating functions. Our approach is algorithmic and, in principle, allows us to enumerate any such class in particular (assuming sufficient computational resources). We give examples to illustrate the method on familiar objects.

Ever since their introduction in [12], permutation grid classes have been appearing in enumerative and structural work on permutation classes and related areas. For a comprehensive introduction to grid classes and their various uses see e.g. [8]. A prominent application of grid classes has been in enumerating other permutation classes, as witnessed by [2, 3, 4, 5, 6, 8, 9, 11].

Ever since their introduction in [12], permutation grid classes have been appearing in enumerative and structural work on permutation classes and related areas. For a comprehensive introduction to grid classes and their various uses see e.g. [8]. A prominent application of grid classes has been in enumerating other permutation classes, as witnessed by [2, 3, 4, 5, 6, 8, 9, 11].

Therefore, it would be convenient to be able to enumerate grid classes themselves. Previous work in this direction includes [1, 7, 8]. Additionally, in [10], we gave explicit generating functions of all juxtapositions of a Catalan class with a monotone class ($n = 2$ case with the context-free class being Catalan). In this talk we extend our previous work to larger and more general grid classes of the form shown in Figure 1.

$\mathcal{M}_1$		$\cdots$		$\mathcal{M}_k$	$\mathcal{C}$	$\mathcal{M}_{k+1}$	$\cdots$	$\mathcal{M}_{k+l}$

Figure 1: All $\mathcal{M}_i$ cells are monotone and $\mathcal{C}$ is context-free.

In what follows we set $n - 1 = k + \ell$ (for some $k, \ell \in \mathbb{Z}_{\geq 0}$) such that $1 \times k$ monotone grid class is juxtaposed on the left of $\mathcal{C}$ and $1 \times \ell$ monotone grid class is juxtaposed on the right of $\mathcal{C}$. Together with $\mathcal{C}$, this gives as $1 \times n$ grid class.

Theorem 1. *Let $\mathcal{C}$ be a context-free permutation class that admits a combinatorial specification which tracks both the right-most and the left-most points. Let $\mathcal{M}_i$ be a sequence of $n - 1$ monotone permutation classes. Then $\mathcal{M}_1|\ldots|\mathcal{M}_k|\mathcal{C}|\mathcal{M}_{k+1}|\ldots|\mathcal{M}_{k+\ell}$ is a context-free permutation class that admits an algebraic generating function.*

Observe that a permutation class $\mathcal{C}$ with finitely many simple permutations is context-free, yielding a notable special case of Theorem 1. This is a strict special case, however, as "Av(321) next to Av(21)" shows (infinitely many simples yet context-free).

Consider a $1 \times n$ grid class

$$\mathcal{M}_1|\ldots|\mathcal{M}_k|\mathcal{C}|\mathcal{M}_{k+1}|\ldots|\mathcal{M}_{k+\ell}.$$

A well-known but key ingredient to our approach is the Chomsky-Schutzenberger theorem which states that context-free combinatorial classes admit algebraic generating functions. The remainder of our effort is then directed at proving that

$$\mathcal{M}_1|\ldots|\mathcal{M}_k|\mathcal{C}|\mathcal{M}_{k+1}|\ldots|\mathcal{M}_{k+\ell}$$

is context-free whenever $\mathcal{C}$ is. Our method makes alterations to the combinatorial specification of $\mathcal{C}$ in such a way that the resulting combinatorial specification represents

$$\mathcal{M}_1|\ldots|\mathcal{M}_k|\mathcal{C}|\mathcal{M}_{k+1}|\ldots|\mathcal{M}_{k+\ell}.$$

The alterations consist of decorating the combinatorial specification of $\mathcal{C}$ with layers of monotone sequences on the right and on the left, as required. This is made systematic through operators which represent various forms/stages of decorations while preserving the character of the combinatorial specification that we started with. Given the "context-freeness" invariant that stays true after applying our operators, we can exploit the inductive nature of the problem to prove Theorem 1.

If time permits, we will present one of the concrete examples below for illustration.

- $\mathrm{Av}(321)|\,\mathrm{Av}(21)$, infinitely many simples in $\mathrm{Av}(321)$
- $\mathrm{Av}(21)|\,\mathrm{Av}(21)|\,\mathrm{Av}(21)$, $n > 2$
- $\mathcal{S}|\,\mathrm{Av}(21)$, finitely many simples in $\mathcal{S}$

where $\mathcal{S}$ denotes the class of separable permutations. Each of the examples demonstrates a different case where our method is applicable. All three examples are worked out explicitly in a thesis draft at

`https://github.com/jsliacan/thesis`

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Erik Slivken University of Paris Diderot, LPSM

This talk is based on joint work with Christopher Hoffman and Douglas Rizzolo

Let $\mathfrak{S}_n(k)$ denote the set of permutations that avoid a decreasing sequence of length $k+1$. We study a uniformly random element of this set of permutations by comparing it with a random walk in a cone. To use this comparison we associate with each permutation $\sigma \in \mathfrak{S}_n(k)$ an ordered k-tuple, P_σ, of continuous functions on $[0,1]$. P_σ is constructed by partitioning the points in the plot of σ into k disjoint increasing sequences $A^j(\sigma)$ where $i \in A^j(\sigma)$ if the longest decreasing sequence in σ that terminates at (i, σ_i) has length j. Linearly interpolating $(0,0)$ and $(1,0)$ with the points

$$s^j(\sigma) = \left(\frac{i}{n}, \frac{\sigma_i - i}{\sqrt{n}}\right)_{i \in A^j(\sigma)}$$

for each j produces the collection P_σ. We now give a definition necessary for the statement of our main result.

Definition 1. Let $\{X_i(t)\}_{i=1}^k$ be standard Brownian bridges conditioned so that $\sum_{i=1}^k X_i = 0$. Let $\{Z_{ij}\}_{1 \le i < j \le k}$ be independent standard complex Brownian bridges. We use these random variables to define the following one parameter family of Hermitian matrices $M(t)$ entry wise as

$$M_{ij}(t) = \begin{cases} X_i(t), & i = j \\ Z_{ij}(t), & i < j \\ \overline{Z_{ij}(t)} & j < i \end{cases} .$$

For any $t \in [0,1]$ let $\lambda_k(t) \ge \cdots \lambda_2(t) \ge \lambda_1(t)$ be the eigenvalues of $M(t)$. We then define

$$TDBM(t) = (\lambda_1(t), \lambda_2(t), \cdots, \lambda_k(t)).$$

Theorem 2. *Let $(P_\sigma(t) | \sigma \in \mathfrak{S}_n(k))$ be with σ chose uniformly at random. Then*

$$(P_\sigma(t) | \sigma \in \mathfrak{S}_n(k)) \longrightarrow TDBM(t).$$

The proof of this theorem follows from coupling P_σ with a random walk in a cone which has a scaling limit of $TDBM(t)$. Let $(X, Y) \in [k]^n \times [k]^n$ and for $j \in [k]$ define

$$a_m^j = \sum_{l=1}^m \mathbf{1}_{x_l = j}, \quad b_m^j = \sum_{l=1}^m \mathbf{1}_{y_l = j} \quad \text{and} \quad c_m^j = a_m^j - b_m^j.$$

Let

$$\Omega_n = \left\{ (X, Y) \in [k]^n \times [k]^n : c_n^j = 0 \ \forall\, j \in [k] \right\}.$$

Considering the sequence $s = (c_m^1, \cdots, c_m^k)_{m=0}^n$ we associate with $\omega \in \Omega_n$ a walk in $\mathbb{Z}^k$ where the steps of the random walk consist of k-tuples where there is exactly one 1, one -1 and the rest are zeros, or k-tuples where every entry is 0. A uniform element of Ω_n produces a random walk, S, where the probably of every entry being zero is $1/k$ and each of the other $k(k-1)$ steps has probability $1/k^2$.

For $\sigma \in \mathfrak{S}_n(k)$ we can construct $\omega_\sigma \in \Omega_n$ by projecting the labels of the points (i, σ_i) used to construct P_σ on the horizontal and vertical axis. Consequently a uniformly random element of $\mathfrak{S}_n(k)$ produces another measure on walks in Ω_n. Our cone of interest is

$$\mathcal{C} := \{(x_1, \cdots, x_k) \in \mathbb{Z}^k | x_1 \leq x_2 \cdots \leq x_k\}.$$

Let S denote a walk in Ω_n that is conditioned to remain in $\mathcal{C}$ and let S' denote a walk in Ω_n obtained from a uniform $\sigma \in \mathfrak{S}_n(k)$. We show there is a coupling of S and S' such that the two walks agree with high probability on the bulk of the walk. We then show a appropriately scaled version of S converges to $TDBM(t)$ and hence so does S'.

Fixed Points

The spacing of the components in $TDBM(t)$ in an open set containing $t = 1/2$ allows us to say something about the distribution of the number and location of the fixed points of permutations in $\widehat{\mathfrak{S}_n(k)}$. Define the following normalized point process that encodes the number and location of fixed points of $\tau \in S_n$:

$$M_{FP,n}(\tau) = \sum_{l=1}^{n} \mathbf{1}_{\tau(i)=i} \delta_{(i-(n+1)/2)/\sqrt{n}}.$$

Let $(\Lambda_1, \cdots, \Lambda_k)$ have joint distribution given by $TDBM(1/2)$ and let $G_1, \cdots G_k$ be independent Bin(1/2k) that are also independent of $(\Lambda_1, \cdots, \Lambda_k)$. We have another point process given by

$$M_{TDBM} = \sum_{j=1}^{k} \mathbf{1}_{\Lambda_j} G_j.$$

Theorem 3. *If $\tau \in \widehat{\mathfrak{S}_n(k)}$ is chosen uniformly at random, then $M_{FP,n}(\tau)$ converges in distribution to M_{TDBM}.*

Stack Sorting r-Tiers

Rebecca Smith SUNY Brockport

This talk is based on joint work with Toufik Mansour and Howard Skogman

Knuth [1] showed that a permutation π can be sorted by a stack (meaning that by applying push and pop operations to the sequence of entries $\pi(1),\ldots,\pi(n)$ we can output the sequence $1,\ldots,n$) if and only if π avoids the permutation 231, i.e., if and only if there do not exist three indices $1 \le i_1 < i_2 < i_3 \le n$ such that $\pi(i_1), \pi(i_2), \pi(i_3)$ are in the same relative order as 231.

Using similar priorities as in our previous work [2], we consider the number of *r-passes* a permutation needs to take through a stack if we only pop the appropriate output values and start over with the remaining entries in the order they would be when they were popped out, i.e. the reverse of their previous order. We define a permutation π to be k-reverse-pass sortable if π is sortable using k r-passes. Permutations that are 1-reverse-pass sortable are simply the stack sortable permutations as defined by Knuth.

Example 1. We show the sorting of $\pi = 354126$ using three r-passes through a stack in Figure 1.

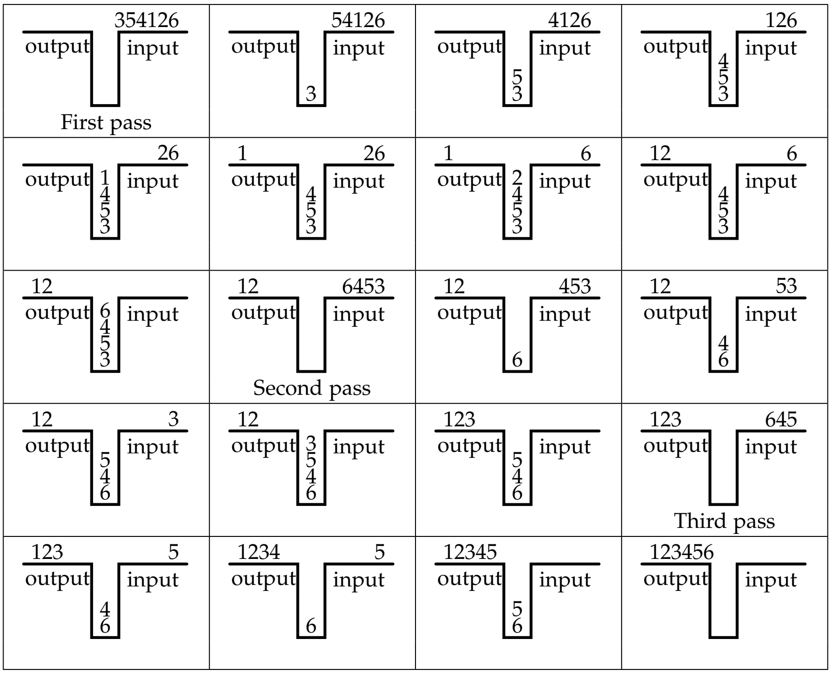

Figure 1: Sorting the permutation 354126 with $k = 3$ passes through a stack.

The *r-tier* of a permutation π, denoted $\rho(\pi)$, will be defined to be the minimum number of r-passes *after* the first r-pass required to sort π. As with the original tier, we can determine the r-tier of a permutation by counting specific covincular patterns.

Theorem 2. *A permutation π is* 2*-reverse-pass sortable, i.e.* $\rho(\pi) \leq 1$, *if and only if π avoids*

$$2413, 2431, 23154.$$

Applying a further restriction to the class of r-tier permutations, namely r-tier relative to length, we find a natural bijection with down/up permutations counted by the Euler numbers.

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Thresholds of Growth Rates for Sum-Closed Classes

Justin Troyka Dartmouth College

Recent work has focused on properties of the set of growth rates of permutation classes, including "thresholds" of growth rates above which a class may have a certain property. For instance, Pantone and Vatter [4] give a number ξ below which there are only countably many growth rates, and they show that ξ is the largest such number. This ξ is defined as the unique positive root of $x^5 - 2x^4 - x^2 - x - 1$, and it is approximately 2.305. Their investigation prominently features sum-closed classes; a class $\mathcal{C}$ is sum-closed if $\sigma, \tau \in \mathcal{C}$ implies $\sigma \oplus \tau \in \mathcal{C}$.

In this presentation, we take Pantone and Vatter's work as a starting point for a deeper investigation into sum-closed classes. We show some general results involving growth rates of sum-closed classes, we highlight a few results on sum-closed classes that are found in Pantone and Vatter's work or follow easily from it, and we give preliminary findings on new kinds of growth-rate thresholds that are a little bit larger than ξ. Many of the ideas and results presented here are from [5].

Given a sum-closed class $\mathcal{C}$, let $\mathcal{C}^{\not\oplus}$ (resp. $\mathcal{C}_n^{\not\oplus}$) denote the set of sum-indecomposable permutations in $\mathcal{C}$ (resp. $\mathcal{C}_n$). Given a set of permutations R, let $\bigoplus R$ denote the smallest sum-closed class that contains every element of R. If $A(x)$ is the ordinary generating function for $\mathcal{C}$, and $C(x)$ for $\mathcal{C}^{\not\oplus}$, then $A(x)$ and $C(x)$ are related by the equation $A(x) = 1/(1 - C(x))$. Since $\mathcal{C}$ is sum-closed, $\mathrm{gr}(\mathcal{C})$ exists, essentially due to Arratia [2].

General result on growth rates of sequences

As mentioned above, when $\mathcal{C}$ is sum-closed, the generating functions for $\mathcal{C}$ and $\mathcal{C}^{\not\oplus}$ are related. The next result shows that, if we add any positive number to $\mathcal{C}_k^{\not\oplus}$ (for at least one k), then $\mathrm{gr}(\mathcal{C})$ increases, and in fact this holds in great generality:

Proposition 1. *Let $C^{(1)}(x) = \sum_{n\geq 1} c_n^{(1)} x^n$ and $C^{(2)}(x) = \sum_{n\geq 1} c_n^{(2)} x^n$ be two formal power series with non-negative real coefficients and zero constant term. For $i \in \{1,2\}$, define*

$$A^{(i)}(x) = \frac{1}{1 - C^{(i)}(x)},$$

and let $a_n^{(i)}$ be the coefficient of x^n in $A^{(i)}(x)$. Assume that $\mathrm{gr}(a_n^{(1)}) < \infty$. If $c_n^{(1)} \leq c_n^{(2)}$ for all n and $c_n^{(1)} < c_n^{(2)}$ for some n, then $\mathrm{gr}(a_n^{(1)}) < \mathrm{gr}(a_n^{(2)})$.

In the talk, we will sketch the proof, which uses complex analysis. This proposition can be useful in dealing with the growth rate of $\mathcal{C}$ based on the sequence $\left(\mathcal{C}_n^{\not\oplus}\right)_{n\geq 1}$, because it gives us strict inequalities of growth rates where otherwise we would have weak inequalities.

Sum-closed classes with growth rate $\leq \xi$

The following result is implicit in Pantone and Vatter [4]:

Proposition 2 (see [4, Sec. 5 & 6])**.** *The number ξ is the smallest that satisfies: if $\mathcal{C}$ is sum-closed and* $\mathrm{gr}(\mathcal{C}) \leq \xi$*, then* $|\mathcal{C}_n^{\not\oplus}| \leq 5$ *for all n.*

From the table of possible sequences $\left(|\mathcal{C}_n^{\not\oplus}|\right)_{n\geq 1}$ given in [4], we also obtain:

Proposition 3. *The number ξ is the smallest that satisfies: if $\mathcal{C}$ is sum-closed and* $\mathrm{gr}(\mathcal{C}) \leq \xi$*, then $\mathcal{C}$ has a rational generating function.*

In contrast, if $\mathcal{C}$ is not required to be sum-closed, then by [1] there are classes with growth rate $\kappa \approx 2.206$ whose generating functions are not rational (and indeed are not even D-algebraic), and κ is the smallest such number.

The threshold of unbounded indecomposables

What is the smallest possible growth rate of a sum-closed class whose indecomposables are unbounded? We provide an example that may have the smallest growth rate.

Let $R = \{1\} \cup \{(12\ldots i) \ominus (12\ldots j) \colon i, j \geq 1\}$, and let $\mathcal{C} = \bigoplus R$. We have $\mathcal{C}^{\not\oplus} = R$, and it turns out that $\mathcal{C} = \mathrm{Av}(321, 3142, 2413)$. An example of a permutation in this class is shown in Figure 1. We also have $|\mathcal{C}_n^{\not\oplus}| = n - 1$ for $n \geq 2$, and $\mathrm{gr}(\mathcal{C})$ is the unique positive root of $x^3 - 3x^2 + 2x - 1$, which is approximately 2.32472. Call this number τ.

Conjecture 4. *τ is the smallest possible growth rate of a sum-closed class $\mathcal{C}$ with the property that $|\mathcal{C}_n^{\not\oplus}|$ is unbounded.*

Our example above shows that τ is a possible growth rate, so the content of Conjecture 4 is that no smaller growth rate is possible. In the other direction, we know from

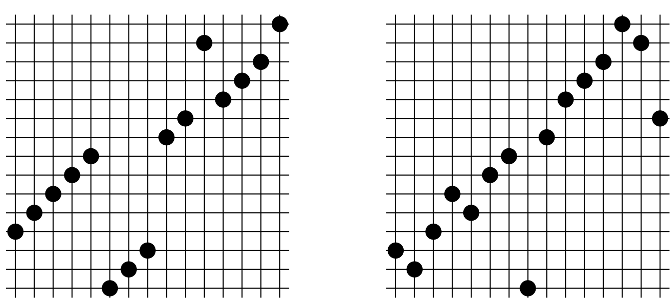

Figure 1: At left, a permutation in $\mathrm{Av}(321, 3142, 2413)$. At right, a permutation in $\mathrm{Av}(312, 4321, 3421)$.

Proposition 2 that all such growth rates are greater than ξ. Thus, the smallest threshold above which $|\mathcal{C}_n^{\not\oplus}|$ can be unbounded is somewhere in the interval $[\xi, \tau]$, whose width is about 0.02.

Conjecture 4 would follow from this conjecture:

Conjecture 5. *If $\mathcal{C}$ is sum-closed and $|\mathcal{C}_n^{\not\oplus}|$ is unbounded, then $|\mathcal{C}_n^{\not\oplus}| \geq n-1$ for all n.*

Conjecture 5 would mean that, for each n, the smallest possible value of $|\mathcal{C}_n|$ is achieved by our example above.

The threshold of exponential indecomposables

What is the smallest possible growth rate of a sum-closed class whose indecomposables are exponential? We provide an example that may have the smallest possible growth rate, but we have not yet investigated thoroughly enough to make a conjecture.

Let $S = \{\sigma \ominus 1 \colon \sigma \in \bigoplus\{1, 21\}\}$, and let $\mathcal{C} = \bigoplus S$. We have $\mathcal{C}^{\not\oplus} = S$, and it turns out that $\mathcal{C} = \mathrm{Av}(312, 4321, 3421)$. An example of a permutation in this class is shown in Figure 1. We have $\mathrm{gr}(\mathcal{C}^{\not\oplus}) = \phi = \frac{1+\sqrt{5}}{2} \approx 1.61803$, which is greater than 1; we also have $\mathrm{gr}(\mathcal{C}) = 1 + \sqrt{2} \approx 2.41421$.

We know from Proposition 2 that $\mathrm{gr}(\mathcal{C}) > \xi$ for all such classes. Thus, the smallest threshold above which $\overline{\mathrm{gr}}(\mathcal{C}^{\not\oplus})$ can be greater than 1 is in the interval $[\xi, 1+\sqrt{2}]$, whose width is about 0.11.

Further questions

- There are no classes with upper or lower growth rate strictly between 1 and ϕ [3], but is there sum-closed $\mathcal{C}$ with $\overline{\mathrm{gr}}(\mathcal{C}^{\not\oplus})$ strictly between 1 and ϕ?
- It is still not known whether every class has a proper growth rate. Given a class $\mathcal{C}$, must $\mathcal{C}^{\not\oplus}$ have a proper growth rate? Neither question's resolution would imply the other; we suspect that the question about $\mathcal{C}^{\not\oplus}$ would be easier, because it suffices to consider sum-closed $\mathcal{C}$, which is much better-behaved than a general class $\mathcal{C}$.

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A Domino Tableau-Based View on Type B Schur-Positivity

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This talk is based on joint work with Alina R. Mayorova

Over the past years, major attention has been drawn to the question of identifying Schur-positive sets, i.e. sets of permutations whose associated quasisymmetric function is symmetric and can be written as a non-negative sum of Schur symmetric functions. The set of permutations avoiding the patterns σ in S_4 such that $|\sigma(1) - \sigma(2)| = 2$, known as arc permutations is one of the most noticeable examples. This paper introduces a new type B extension of Schur-positivity based on Chow's quasisymmetric functions and generating functions for domino tableaux. In particular we design descent preserving bijections between signed arc permutations and sets of domino tableaux to show that they are indeed type B Schur-positive.

Background

Young tableaux and descent sets

A **partition** λ of an integer n, denoted $\lambda \vdash n$ is a sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_p)$ of $\ell(\lambda) = p$ parts sorted in decreasing order such that $|\lambda| = \sum_i \lambda_i = n$. A partition λ is usually represented as a Young diagram of $n = |\lambda|$ boxes arranged in $\ell(\lambda)$ left justified rows so that the i-th row from the top contains λ_i boxes. A Young diagram whose boxes are filled with positive integers such that the entries are increasing along the rows and strictly increasing down the columns is called a **semistandard Young tableau**. If the entries of a semistandard Young tableau are restricted to the elements of $[n]$ and strictly increasing along the rows, we call it a **standard Young tableau**. The partition λ is the **shape** of the tableau and we denote $SYT(\lambda)$ (resp. $SSYT(\lambda)$) the set of standard (resp. semistandard) Young tableaux of shape λ. Define the **descent set of a standard Young tableau** T as $Des(T) = \{1 \leq i \leq n-1 \mid i \text{ is in a strictly higher row than } i+1\}$. Similarly the descent set of a permutation π in S_n is the subset of $[n-1]$ defined as $Des(\pi) = \{1 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}$.

Schur-positivity

Let $X = \{x_1, x_2, \cdots\}$ be a totally ordered set of commutative indeterminates. Given any subset $\mathcal{A}$ of permutations in S_n, Gessel introduces in [6] the formal power series in $\mathbb{C}[X]$:

$$Q(\mathcal{A})(X) = \sum_{\pi \in \mathcal{A}} F_{Des(\pi)}(X),$$

where for any subset $I \subseteq [n-1]$, $F_I(X)$ is the **fundamental quasisymmetric function** defined by:

$$F_I(X) = \sum_{\substack{i_1 \leq \cdots \leq i_n \\ k \in I \Rightarrow i_k < i_{k+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

The power series $F_I(X)$ is not symmetric in X but verifies the property that for any strictly increasing sequence of indices $i_1 < i_2 < \cdots < i_p$ the coefficient of $x_1^{k_1} x_2^{k_2} \cdots x_p^{k_p}$ is equal to the coefficient of $x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_p}^{k_p}$. In [7] Gessel and Reutenauer looked at the problem of characterising the sets $\mathcal{A}$ for which $Q(\mathcal{A})$ is symmetric. Further the question of determining **Schur-positive** sets, i.e. the sets $\mathcal{A}$ for which $Q(\mathcal{A})$ can be expanded with non-negative coefficients in the Schur basis received significant attention. Classical examples of Schur-positive sets include inverse descent classes, Knuth classes [6] and conjugacy classes [7]. As a more sophisticated example, Elizalde and Roichman proved [3] the Schur-positivity of **arc permutations**, i.e the set of permutations in S_n avoiding the patterns σ in S_4 such that $|\sigma(1) - \sigma(2)| = 2$. Arc permutations are alternatively defined as the set of permutations π in S_n such that for any $1 \leq j \leq n$, $\{\pi(1), \pi(2), \cdots, \pi(j)\}$ is an interval in $\mathbb{Z}_n$. Other advanced examples of Schur-positive sets can be found in [5]. Many of these results are the consequence of two main facts.

1. Denote s_λ the Schur symmetric functions indexed by $\lambda \vdash n$. s_λ is the generating function for semistandard Young tableaux of shape λ. It follows (see e.g. [9, 7.19.7]) that

$$s_\lambda(X) = \sum_{T \in SSYT(\lambda)} X^T = \sum_{T \in SYT(\lambda)} F_{Des(T)}(X). \tag{1}$$

2. There are various descent preserving bijections relating sets of permutations and standard Young tableaux, e.g. the celebrated *Robinson-Schensted* (RS) correspondence. The proof in [3] also uses such a bijection between arc permutations and standard Young tableaux of shapes $(n-k, 1^k)$ and $(n-k-2, 2, 1^k)$.

A **type B extension** of Schur-positivity deals with B_n, the **hyperoctahedral group** of order n instead of S_n. B_n is composed of all permutations π on

$$\{\text{-}n, \cdots, \text{-}2, \text{-}1, 0, 1, 2, \cdots, n\}$$

such that $\pi(-i) = -\pi(i)$ for all i. Such permutations usually referred to as **signed permutations** are fully described by their restriction to $[n]$. To extend items 1 and 2 above, two options are available and depend on the definition for the descent of signed permutations. In [1], Adin et al. use the notion of **signed descent set**, i.e. the couple (S, ϵ) defined for $\pi \in B_n$ as $S = \{n\} \cup \{1 \leq i \leq n-1 \mid \pi(i) > 0 \text{ and } \pi(i) > \pi(i+1) \text{ or } \pi(i) < 0 \text{ and either } \pi(i+1) > 0 \text{ or } |\pi(i)| > |\pi(i+1)|\}$ and ϵ is the mapping from S to $\{-, +\}$ defined as $\epsilon(s) = +$ if $\pi(s) > 0$ and $\epsilon(s) = -$, otherwise. There is a signed descent preserving analogue of the RS correspondence relating signed permutations and *bi-tableaux*, i.e. couples of Young tableaux with specific constraints and

[1] proves an analogue of Equality (1) between their generating function and *Poirier's signed quasisymmetric functions*. While the authors succeed in extending most of the results known in type A, providing another framework relying on a more intuitive definition of descent for signed permutations appears as a natural question. Indeed one can simply define the descent set of $\pi \in B_n$ as the subset of $\{0\} \cup [n-1]$ equal to $Des(\pi) = \{0 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}$. The descent preserving analogue of the RS correspondence in this case relates signed permutations and *domino tableaux* (see next section). We use the generating function for domino tableaux and *Chow's type B quasisymmetric functions* to develop this alternative type B extension of Schur-positivity. Moreover, we introduce a new descent preserving bijection between *signed arc permutations* and domino tableaux to show that the former set is type B Schur-positive according to the definition of descent stated above.

A new definition of type B Schur-positivity based on Chow's quasisymmetric functions and domino functions

Domino tableaux and Chow's type B quasisymmetric functions

For $\lambda \vdash 2n$, a **standard domino tableau** T of shape λ is a Young diagram of shape $shape(T) = \lambda$ tiled by **dominoes**, i.e. 2×1 or 1×2 rectangles filled with the elements of $[n]$ such that the entries are strictly increasing along the rows and down the columns. In the sequel we consider only the set $\mathcal{P}^0(n)$ of *empty* 2*-core partitions* $\lambda \vdash 2n$ that fit such a tiling. A standard domino tableau T has a descent in position $i > 0$ if $i+1$ lies strictly below i in T and has descent in position 0 if the domino filled with 1 is vertical. We denote $Des(T)$ the set of all its descents. A **semistandard domino tableau** T of shape $\lambda \in \mathcal{P}^0(n)$ and weight $w(T) = \mu = (\mu_0, \mu_1, \mu_2, \cdots)$ with $\mu_i \geq 0$ and $\sum_i \mu_i = n$ is a tiling of the Young diagram of shape λ with horizontal and vertical dominoes labelled with integers in $\{0, 1, 2, \cdots\}$ such that labels are non decreasing along the rows, strictly increasing down the columns and exactly μ_i dominoes are labelled with i. If the top leftmost domino is vertical, it cannot be labelled 0. Denote $SDT(\lambda)$ (resp. $SSDT(\lambda)$) the set of standard (resp. semistandard) domino tableaux of shape λ.

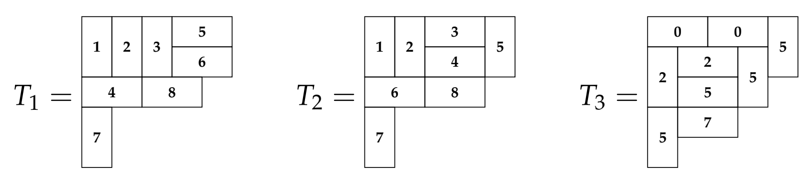

Figure 1: Two standard domino tableaux T_1 and T_2 of shape $(5,5,4,1,1)$ and descent set {0,3,5,6} and a semistandard tableau T_3 of shape $(5,5,4,3,1)$ and weight $\mu = (2,0,2,0,0,4,0,1)$.

Chow defines in [2] an analogue of Gessel's algebra of quasisymmetric functions that is dual to the Solomon's descent algebra of type B. Let $X = \{x_0, x_1, \cdots, x_i, \cdots\}$ be a set of totally ordered commutative indeterminates and I be a subset of $\{0\} \cup [n-1]$, he defines a type B analogue of the fundamental quasisymmetric functions

$$F_I^B(X) = \sum_{\substack{0=i_0 \leq i_1 \leq i_2 \leq \ldots \leq i_n \\ j \in I \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \ldots x_{i_n}.$$

Note the particular rôle of the variable x_0. Given a semistandard domino tableau T of weight μ, denote X^T the monomial $x_0^{\mu_0} x_1^{\mu_1} x_2^{\mu_2} \cdots$. In [8], we introduce a variant of the generating function $\mathcal{G}_\lambda$ for semistandard domino tableaux of shape $\lambda \in \mathcal{P}^0(n)$ taking into account the zero values called **domino function** and show that it is related with type B fundamental quasisymmetric functions through:

$$\mathcal{G}_\lambda(X) = \sum_{T \in SSDT(\lambda)} X^T = \sum_{T \in SDT(\lambda)} F^B_{Des(T)}. \tag{2}$$

Type B Schur positivity

Similarly to the type A case, given any subset $\mathcal{B}$ of B_n we look at the Chow's type B quasisymmetric function

$$Q(\mathcal{B})(X) = \sum_{\pi \in \mathcal{B}} F^B_{Des(\pi)}(X).$$

Definition 1. We say that a set $\mathcal{B} \subset B_n$ is **type B Schur positive** or **$\mathcal{G}$-positive** if the function $Q(\mathcal{B})$ can be written as a non-negative sum of domino functions.

Remark 2. Let $X^* = X \setminus \{x_0\}$. According to our definition of type B Schur positivity, the fact that the quasisymmetric function $Q(\mathcal{B})$ is $\mathcal{G}$-positive does not imply that it belongs to the set of symmetric functions $\Lambda_n[X]$. Rather it belongs to the vector space $\Lambda_n^B[X]$ spanned by the functions of the form $x_0^k f(X^*)$ where f is any symmetric function of $\Lambda_{n-k}[X^*]$. Namely, if $Q(\mathcal{B})$ is $\mathcal{G}$-positive then

$$Q(\mathcal{B}) \in \Lambda_n^B[X] = \sum_{k=0}^{n} x_0^k \Lambda_{n-k}[X^*].$$

Remark 3. The set of domino functions $\{\mathcal{G}_\lambda\}_{\lambda \in \mathcal{P}^0(n)}$ is not linearly independent and is not a basis of $\Lambda_n^B[X]$. As a result, there might be more than one way to decompose a function in $\Lambda_n^B[X]$ as a sum of domino functions. However, we have the two following remarks.

- Using a well known bijection between partitions λ in $\mathcal{P}^0(n)$ and pairs of partition (λ^-, λ^+) such that $|\lambda^-| + |\lambda^+| = n$. One can show that the subfamily $\{\mathcal{G}_{\lambda^-,(k)}\}_{k \leq n, \lambda^- \vdash n-k}$ is a basis of $\Lambda_n^B[X]$.
- Domino functions can be expanded as a non-negative sum of the $s^B_{(n-|\rho|,\rho)}(X) = x_0^{n-|\rho|} s_\rho(X^*)$ $(|\rho| \leq n)$ which is a natural basis of $\Lambda_n^B[X]$. As a result, if a function $Q(\mathcal{B})$ is $\mathcal{G}$-positive, then it can also be decomposed with non-negative coefficients in the basis $s^B_{(n-|\rho|,\rho)}$.

We proceed with our first examples of $\mathcal{G}$-positive sets.

Proposition 4. *The inverse descent sets* $D_{n,J}^{B,-1} = \{\pi \in B_n \mid Des(\pi^{-1}) = J\}$ *are* $\mathcal{G}$*-positive.*

Proof. According to the type B analogue of the RS-correspondence, there is a descent preserving bijection between signed permutations π of B_n and couples of standard domino tableaux (P_π, Q_π) such that $shape(P_\pi) = shape(Q_\pi) \in \mathcal{P}^0(n)$. We have $Des(\pi) = Des(Q_\pi)$ and $Des(\pi^{-1}) = Des(P_\pi)$. It follows

$$Q(D_{n,J}^{B,-1}) = \sum_{\substack{\pi \in B_n \\ Des(\pi^{-1})=J}} F_{Des(\pi)} = \sum_{\lambda \in \mathcal{P}^0(n)} \sum_{\substack{P \in SDT(\lambda) \\ Des(P)=J}} \sum_{Q \in SDT(\lambda)} F^B_{Des(Q)}.$$

The proof follows from Equation (2). □

Using the same bijection between signed permutation and couples of domino tableaux, another important example of $\mathcal{G}$-positive set is type B Knuth classes. Given a standard domino tableau T we denote the corresponding type B Knuth class $\mathcal{C}_T = \{\pi \in B_n \mid P_\pi = T\}$.

Proposition 5. *Let* $\lambda \in \mathcal{P}^0(n)$ *and* $T \in SDT(\lambda)$*. The type B Knuth class* $\mathcal{C}_T$ *is* $\mathcal{G}$*-positive.*

Proof. Compute $Q(\mathcal{C}_T)$ using the type B analogue of the RS-correspondence.

$$Q(\mathcal{C}_T) = \sum_{\substack{\pi \in B_n \\ P_\pi = T}} F^B_{Des(\pi)} = \sum_{Q \in SDT(\lambda)} F^B_{Des(Q)} = \mathcal{G}_\lambda.$$

Application to signed arc permutations

A permutation $\pi \in B_n$ is called **a signed arc permutation** if for $1 \leq i \leq n$ the set $\{|\pi(1)|, |\pi(2)|, \cdots |\pi(i-1)|\}$ is an interval in Z_n and $\pi(i) > 0$ if $|\pi(i)| - 1 \in \{|\pi(1)|, |\pi(2)|, \cdots |\pi(i-1)|\}$ and $\pi(i) < 0$ otherwise. The set of signed arc permutations is denoted by $\mathcal{A}_n^s$. One can prove (see [4]) that signed arc permutation are exactly those permutations of B_n that avoid the following 24 patterns:

$$[1,-2,3], [1,3,2], [2,-3,1], [2,1,3], [3,-1,2], [3,2,1].$$

The main result of this paper follows:

Theorem 6. *The set of signed arc permutations* $\mathcal{A}_n^s$ *is* $\mathcal{G}$*-positive. Moreover,*

$$\sum_{\pi \in \mathcal{A}_n^s} F^B_{Des(\pi)} = \mathcal{G}_{(2n)} + \mathcal{G}_{(2n-1,1)} + \mathcal{G}_{(2n-2,1,1)} + \mathcal{G}_{(2n-3,1,1,1)} + 2 \sum_{a \geq 2n-a \geq 2} \mathcal{G}_{(a,2n-a)} + \sum_{a \geq 2n-a-2 \geq 2} \mathcal{G}_{(a,2n-a-2,2)} + \sum_{a \geq 2n-a-2 \geq 2} \mathcal{G}_{(a,2n-a-2,1,1)}. \quad (3)$$

To prove Theorem 6 we introduce a new descent-preserving bijective map from $\mathcal{A}_n^s$ to the sets of standard domino tableaux with shapes equal to the indices of the domino functions in formula 3.

We start by giving a more precise description of signed arc permutations. Given two sequences of integers A and B, denote $A\ sh\ B$ the set of integers sequence π consisting of all elements of A and B such that A and B form ordered subsequences of π. The following table breaks the set of signed arc permutations into 6 non overlapping types. Types 1 and 2 have only either positive or negative entries. The four remaining types have at least one negative and one positive integer and are characterised according of the sign of their entries with absolute value 1 and n.

Type	Content
1	$\bigcup_{k\in[2,n]}(k\cdots n, 1\cdots k-1)\ \bigcup\ (1\cdots n)$
2	$\bigcup_{k\in[2,n]}(\text{-}(k-1)\cdots\text{-}1, \text{-}n\cdots\text{-}k)\ \bigcup\ (\text{-}n\cdots\text{-}1)$
3	For any k,l such that $n>k>l\geq 1$, $\bigcup_{k,l}(\text{-}k, \text{-}(k-1)\cdots\text{-}(l+1))\ sh\ (k+1\cdots n, 1\cdots l)$
4	For any k,l such that $n>l>k\geq 1$, $\bigcup_{k,l}(\text{-}k, \text{-}(k-1)\cdots\text{-}1, \text{-}n\cdots\text{-}(l+1))\ sh\ (k+1\cdots\cdots l)$
5	$\bigcup_{k\in[1,n-1]}(\text{-}k, \text{-}(k-1)\cdots\text{-}1)\ sh\ (k+1,\cdots n)$
6	$\bigcup_{k\in[1,n-1]}(\text{-}n\cdots\text{-}(k+1))\ sh\ (1\cdots k)$

The next step is to build the bijections with standard domino tableaux. We start with types 5 and 6.

Proposition 7. *Both type 5 and type 6 permutations are in descent-preserving bijection with the set of standard domino tableaux of shapes $(a, 2n-a)$ for a such that $a \geq 2n-a \geq 2$.*

Proof. We give the proof for type 5 permutations but the same reasoning applies to type 6. Firstly, map $\pi^0 = (\text{-}1,\ 2\cdots n)$ to the standard domino tableau T_0 composed of n vertical dominoes. We have $Des(\pi^0) = Des(T_0) = \{0\}$. Secondly, let $\pi = \pi_1\pi_2\cdots\pi_n \neq \pi^0$. We build recursively a two-row standard domino tableau according to the following procedure. At step $1 \leq i \leq n$ we add a domino with label i.

- if $\pi_i > 0$, add a horizontal domino on the first row in rightmost position.
- if $\pi_i < 0$, we add either a horizontal domino in the second row or a vertical domino across the two rows in rightmost position. As the difference of lengths is even, only one of these two positions are available.

The construction is clearly bijective and descent preserving. Indeed, there is a descent at $i > 0$ in the tableau if $\pi_i > 0 > \pi_{i+1}$ and at $i = 0$ if $\pi_1 < 0$, i.e. exactly when we have a descent in type 5 signed arc permutations. □

The two following bijections are modifications of the bijection above and are left to the reader.

Proposition 8. *Type 1 and type 2 permutations together are in descent-preserving bijection with the set of standard domino tableaux of shapes $(2n), (2n-1,1), (2n-2,1,1)$ and $(2n-3,1,1,1)$.*

Proposition 9. *Type 4 signed arc permutations are in descent preserving bijection with standard domino tableaux of shape* $(a, 2n-a-2, 1, 1)$ *for a such that* $a \geq 2n-a-2 \geq 2$.

The type 3 case is more complicated. For any type 3 signed arc permutation π, one should consider seven different cases depending on the existence or not of negative numbers before n, between n and 1 and after 1. All together, these cases lead to the following proposition, which we give here without proof:

Proposition 10. *Type 3 signed arc permutations are in descent preserving bijection with standard domino tableaux of shapes* $(a, 2n-a-2, 2)$ *for a such that* $a \geq 2n-a-2 \geq 2$.

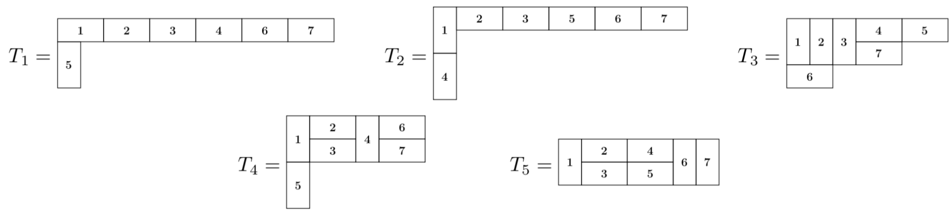

Figure 2: Examples of the bijections above. Tableaux T_1, T_2, T_3, T_4 and T_5 correspond respectively to the permutations (4 5 6 7 1 2 3), (-3 -2 -1 -7 -6 -5 -4), (-4 -3 5 6 7 -2 1), (-3 4 -2 -1 -7 5 -6) and (-5 6 -4 7 -3 -2 -1) of respective types 1, 2, 3, 4 and 5.

One finishes the proof of Theorem 6 using the descent preserving bijections above to write:

$$\sum_{\omega \in A_n^s} F^B_{Des(\omega)} = \sum_{T \in SDT(2n)} F^B_{Des(T)} + \sum_{T \in SDT(2n-2,1,1)} F^B_{Des(T)} + \sum_{T \in SDT(2n-1,1)} F^B_{Des(T)} + \sum_{T \in SDT(2n-3,1,1,1)} F^B_{Des(T)}$$
$$+ \sum_{a, T \in SDT(a, 2n-a-2,2)} F^B_{Des(T)} + \sum_{a, T \in SDT(a,2n-a-2,1,1)} F^B_{Des(T)} + 2 \sum_{a, T \in SDT(a,2n-a)} F^B_{Des(T)}$$

which gives Theorem 1 after application of Equation (2).

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The Substitution Decomposition of Matchings and RNA Secondary Structures

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This talk is based on joint work with Aziza Jefferson

The substitution decomposition (also known as the modular decomposition) has proved to be a versatile tool in the study of a wide variety of combinatorial objects. This concept dates back to a 1953 talk of Fraïssé [6], although its first significant application was in Gallai's 1967 paper [7] (see [12] for a translation). In 1980, Földes [5] described the substitution decomposition in the general context of relational structures, and Möhring and Radermacher [14] surveyed several applications of the approach in 1984. More recently, the substitution decomposition has been applied to enumerative problems in the study of permutation patterns. In particular, Albert and Atkinson [1] established that permutation classes with only finitely many simple permutations have algebraic generating functions.

We develop the theory of the substitution decomposition of matchings. We use the term *matching* as shorthand for a *complete, ordered matching* on $[2n] = \{1, 2, \ldots, 2n\}$, which means our matchings have labeled vertices $[2n]$ and every vertex is incident to precisely one edge. We refer to the number of edges in a matching as the *size* of the matching and thus there are $(2n-1)!!$ matchings of size n.

While one inspiration for this work is the growing interest in the enumeration of pattern-avoiding matchings [2, 4, 3, 8, 9, 10, 11, 13], much of which has been done by researchers from the permutation patterns community, our primary motivation is that the substitution decomposition proves to be an ideal language in which to describe various families of RNA secondary structures. We refer to Reidys' text [15] for an overview of RNA secondary structures in general.

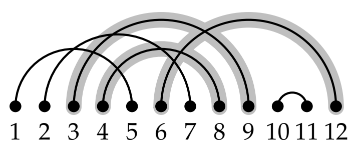

Figure 1: The containment order on matchings.

The matching M is said to *contain* the matching N if one can obtain N from M by deleting edges and relabeling the vertices in the unique order-preserving manner; otherwise we say that M *avoids* N. For example, the shaded edges in the matching in Figure 1 show that this matching contains the matching  of size 3. On the hand, it can be checked that this matching avoids , as it does not contain a nested sequence of three edges.

This order on matchings is connected to the usual permutation pattern order by what

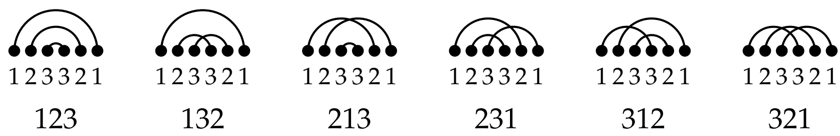

Figure 2: The permutational matchings corresponding to permutations of length three.

we call permutational matchings (following Jelínek [8] for the term but Bloom and Elizalde [2] for the definition). Given a permutation π of length n, the *permutational matching* corresponding to π, which we denote by M_π, is the matching on the vertices $[2n]$ in which vertex i is adjacent to vertex $(2n + 1 - \pi(i))$. Figure 2 shows the permutational matchings of size three.

It is easy to verify that the permutation σ is contained in the permutation π if and only if the matching M_σ is contained in the matching M_π. Thus the study of permutation patterns is the special case of the study of matching patterns that avoid the matching consisting of two non-nesting, non-crossing edges, ∩∩.

Our main result lifts the main enumerative result of Albert and Atkinson [1] to the matching context.

Theorem 1. *Every matching class with only finitely many strongly indecomposable matchings has an algebraic generating function.*

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Homomorphisms on Noncommutative Symmetric Functions and Permutation Enumeration

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Gelfand, Krob, Lascoux, Leclerc, Retakh, and Thibon in their seminal 1995 paper [7] introduced the algebra **Sym** of noncommutative symmetric functions and elucidated its connections to combinatorics, representation theory, Lie algebras, and mathematical physics. However, it is worth noting that noncommutative symmetric functions implicitly appeared earlier in the Ph.D. thesis of Ira Gessel [8] in the context of permutation enumeration. Gessel showed that many permutation enumeration formulas involving descents can be proven by first deriving a lifting of the formula in **Sym** and then applying an appropriate homomorphism. Moreover, he proved a result that we call the "run theorem", which allows one to obtain noncommutative symmetric function formulas for counting permutations with restrictions on the lengths of their increasing runs (i.e., distances between consecutive descents).

In a series of recent papers [10, 17, 18], we further develop this method of permutation enumeration, introducing new homomorphisms and a generalization of the run theorem which allows for a much wider variety of restrictions on run lengths. This machinery is used to provide new derivations of formulas previously obtained by Carlitz [1], Chebikin [2], David–Barton [3], Elizalde [4], Elizalde–Noy [5], Entringer [6], Petersen [11, 12], Remmel [13], Stanley [14, 15], and Stembridge [16], as well as to prove a myriad of new formulas. These include a few results on consecutive pattern avoidance.

The newest development in this domain utilizes the theory of shuffle-compatible permutation statistics [9]. It can be proven that each shuffle-compatible permutation statistic gives rise to a homomorphism on **Sym** which can be used to count permutations by the corresponding "inverse statistic". For example, given a permutation π, let $\operatorname{des}(\pi)$ denote the number of descents of π and let $\operatorname{ides}(\pi)$ denote the number of descents of π^{-1}. Also, let

$$M_{m,n}^{\text{ides}}(t) := \sum_{\pi \in \text{Av}_n(\underline{12\cdots m})} t^{\text{ides}(\pi)+1}$$

denote the polynomial encoding the distribution of the ides statistic over length n permutations avoiding the increasing consecutive pattern $\underline{12\cdots m}$. Then the homomorphism arising from the shuffle-compatibility of the descent number des can be used to obtain the new formulas

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{M_{m,n}^{\text{ides}}(t)}{(1-t)^{n+1}} x^n &= \sum_{k=0}^{\infty} \left[\sum_{j=0}^{\infty} \left(\binom{k+jm-1}{k-1} x^{jm} - \binom{k+jm}{k-1} x^{jm+1} \right) \right]^{-1} t^k \qquad (\heartsuit) \\ &= m \sum_{k=0}^{\infty} \left[\sum_{j=1}^{m-1} \frac{1-\omega^{-j}}{(1-\omega^j x)^k} \right]^{-1} t^k\end{aligned}$$

where $\omega = e^{2\pi i/m}$. Taking the limit of (♡) as $m \to \infty$ and extracting coefficients of x^n recovers the classical identity

$$\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{k=0}^{\infty} k^n t^k$$

for the Eulerian polynomials $A_n(t)$ defined by

$$A_n(t) := \sum_{\pi \in \mathfrak{S}_n} t^{\operatorname{des}(\pi)+1}.$$

In this talk, we give an overview of this method of using homomorphisms on noncommutative symmetric functions in permutation enumeration, survey some of the formulas that can be obtained using this method, and conclude with a brief discussion of possible future directions of work involving applications of this method to consecutive pattern avoidance.

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